Computer Aided Geometric Design 60 (2018) 1–9

Contents lists available at ScienceDirect

Computer Aided Geometric Design

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Cissoid constructions of augmented rational ruled surfaces

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A R T I C L E I N F O

Article history:
Available online 12 December 2017

Keywords:
Rational surface
Ruled surface
Steiner surfaces
Augmented surface
Cissoid
Symbolic computation

A B S T R A C T

Given two real affine rational surfaces we derive a criterion for deciding the rationality of their cissoid. Furthermore, when one of the surfaces is augmented ruled and the other is either an augmented ruled or an augmented Steiner surface, we prove that the cissoid is rational. Furthermore, given rational parametrizations of the surfaces, we provide a rational parametrization of the cissoid.

1. Introduction

The automatic geometric design theory studies certain geometric constructions on surfaces, many of them with origins in classical geometry. Probably the most significant and successful one is the offset (parallel surfaces in the classical language); e.g. Hoschek and Lasser (1993) and Arrondo et al. (1997). Other used constructions are pedal, conchoid, or convolution of surfaces (see Gruber and Peternell, 2013; Peternell et al., 2015; Pottmann et al., 1996; Lávička and Bastl, 2007; Vršek and Lávička, 2010). In this paper, we deal with a different geometric construction, also with origins in former geometric studies, the cissoid construction.

Throughout the paper, we denote by $\mathcal{F}, \mathcal{G} \subset \mathbb{R}^3$ two real affine surfaces. In addition, we denote by F and G the defining polynomials of $\mathcal{F}$ and $\mathcal{G}$ respectively. Furthermore, when the surfaces are rational, we will denote by $\mathbf{f}(u,v)$ and $\mathbf{g}(u,v)$ a parametrization of $\mathcal{F}$ and $\mathcal{G}$, respectively.

In this situation, given a reference point $\mathcal{O} \in \mathbb{R}^3$, and two non-zero numbers $\lambda, \mu \in \mathbb{R}$, we define the *cissoid* of $\mathcal{F}$ and $\mathcal{G}$ as the geometric locus of those points P such that

$$\overline{\mathcal{O}P} = \lambda\,\overline{\mathcal{O}X} + \mu\,\overline{\mathcal{O}Y}$$

for some $X \in \mathcal{F}$ and some $Y \in \mathcal{G}$. We denote the cissoid as $\mathcal{F} \diamond_{\{\lambda,\mu,\mathcal{O}\}} \mathcal{G}$. Let $\mathbf{0} = (0,0,0) \in \mathbb{R}^3$, and τ the translation in $\mathbb{R}^3$ such that $\tau(\mathcal{O}) = \mathbf{0}$. Then, it holds that

$$\mathcal{F} \diamond_{(\lambda,\mu,\mathcal{O})} \mathcal{G} = \tau^{-1}(\tau(\mathcal{F}) \diamond_{(\lambda,\mu,\mathbf{0})} \tau(\mathcal{G})).$$

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https://doi.org/10.1016/j.cagd.2017.12.001
0167-8396/

On the other hand, if we denote by $\lambda\mathcal{F}$ and by $\mu\mathcal{G}$ the affine surfaces defined by the polynomials $F(\frac{x}{\lambda},\frac{y}{\lambda},\frac{z}{\lambda})$ and $G(\frac{x}{\mu},\frac{y}{\mu},\frac{z}{\mu})$, respectively, we have that

$$\mathcal{F}\diamond_{(\lambda,\mu,\mathcal{O})}\mathcal{G}=\lambda\mathcal{F}\diamond_{(1,1,\mathcal{O})}\mu\mathcal{G}.$$

Therefore, without loss of generality, in the sequel, we will consider that $\mathcal{O}=\mathbf{0}$, $\lambda=1$, $\mu=1$. Furthermore, we will simplify the notation writing $\mathcal{F}\diamond\mathcal{G}$ instead of $\mathcal{F}\diamond_{(1,1,\mathbf{0})}\mathcal{G}$.

The second notion that we use in the paper is the concept of an augmented surface. Let us assume that $\mathcal{F}$ is parametrized by $\mathbf{f}(u,v)$, and let $T(u,v)$ be a function. We will call the surface parametrized as $T(u,v)\mathbf{f}(u,v)$ a *T-augmentation of the surface $\mathcal{F}$*. Note that if $\mathbf{f}$ is rational and T is a rational function, then the T-augmentation is a rational surface. When $\mathcal{F}$ is ruled or quadratically parametrized we get augmented ruled or augmented quadratically parametrized surfaces, respectively. We recall that the quadratically parametrizable surfaces are projections of a Veronese surface of degree four, and often denoted as Steiner surfaces. The most famous representative is Steiner's roman surface, see for instance Coffman et al. (1996), Degen (1996). In this paper, we will refer to them as Steiner surfaces.

In this paper, we will deal with the cissoid of two rational algebraic surfaces and we study the rationality, and actual computation of parametrizations, of the cissoid. More precisely, the main contribution in this paper is as follows.

Contribution Let $\mathcal{F},\mathcal{G}$ be rational real affine surfaces. We provide a criterion for deciding the rationality of the cissoid in terms of rationality of (a component of) an auxiliary variety constructed from $\mathcal{F}$ and $\mathcal{G}$. In addition, we prove that, if $\mathcal{F}$ is an augmented ruled and $\mathcal{G}$ is either an augmented ruled or an augmented Steiner surface, the cissoid of $\mathcal{F}$ and $\mathcal{G}$ is a rational surface. Furthermore, given rational parametrizations of $\mathcal{F}$ and $\mathcal{G}$, we provide a rational parametrization of the cissoid.

The paper is structured in three main sections. In Section 2, we show how the theory of ideals and Gröbner basis can be applied to compute the implicit equation of the cissoid. In Section 3, we characterize the rationality of the cissoid by means of the rationality of an auxiliary variety. This result provides indeed a method to check algorithmically the rationality of the cissoid of two rational surfaces. Finally, in Section 4 we analyze the cissoid of augmented ruled surfaces.

2. Computation of the cissoid surface

In this section we deal with the problem of computing the cissoid $\mathcal{F}\diamond\mathcal{G}$. In the first part of the section, we assume that $\mathcal{F}$ and $\mathcal{G}$ are given by means of their defining polynomials, namely F and G, and we show how to determine the implicit representation of the cissoid.

For this purpose, since we have assumed that the reference point is the origin $\mathbf{0}$ of the affine space $\mathbb{R}^3$, the cissoid $\mathcal{F}\diamond\mathcal{G}$ is given by the geometric locus of those points $\mathbf{a}+\mathbf{b}$, for $\mathbf{a}\in\mathcal{F}$ and $\mathbf{b}\in\mathcal{G}$ such that $\{\mathbf{0},\mathbf{a},\mathbf{b}\}$ are collinear. Consequently, it is defined by

$$\mathcal{F}\diamond\mathcal{G}=\{\mathbf{a}+\mathbf{b},\ \text{with}\ \mathbf{a}\in\mathcal{F},\mathbf{b}\in\mathcal{G},\ \text{and}\ \{\mathbf{0},\mathbf{a},\mathbf{b}\}\ \text{collinear}\}. \tag{1}$$

For any point $\mathbf{b}\in\mathcal{G}$ there is typically a finite number of corresponding points $\mathbf{a}_i\in\mathcal{F}$, related to the degree of $\mathcal{F}$. Note that, choosing $\mathcal{G}$ as the sphere of radius d centered at the origin, the cissoid $\mathcal{F}\diamond\mathcal{G}$ becomes the conchoid of $\mathcal{F}$ with respect to the origin and distance d (see e.g. Gruber and Peternell, 2013; Peternell et al., 2015).

We show how to compute the cissoid by means of Gröbner bases. In Albano and Roggero (2010), it is shown how to determine the cissoid by means of resultants, but extraneous factors may appear. Let $\mathbf{a}=(a_1,a_2,a_3)$, $\mathbf{b}=(b_1,b_2,b_3)$, and $\mathbf{x}=(x,y,z)$. To determine the implicit equation of the cissoid $\mathcal{F}\diamond\mathcal{G}$ we consider the ideal

$$\begin{aligned} J &=<\mathbf{x}-\mathbf{a}-\mathbf{b},\mathbf{a}\times\mathbf{b},F(\mathbf{a}),G(\mathbf{b})> \\ &=<x-a_1-b_1,y-a_2-b_2,z-a_3-b_3,a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1, \\ &\qquad F(a_1,a_1,a_3),G(a_1,a_2,a_3)> \end{aligned} \tag{2}$$

in the polynomial ring in $\mathbb{C}[\mathbf{a},\mathbf{b},\mathbf{x}]=\mathbb{C}[a_1,a_2,a_3,b_1,b_2,b_3,x,y,z]$. The last two entries in J ensure that $\mathbf{a}\in\mathcal{F}$ and $\mathbf{b}\in\mathcal{G}$, $\mathbf{a}\times\mathbf{b}$ requires the collinearity of $\mathbf{a},\mathbf{b}$ and the origin, and hence if $\mathbf{x}=\mathbf{a}+\mathbf{b}$ then $\mathbf{x}\in\mathcal{F}\diamond\mathcal{G}$ (see (1)). This implies that the cissoid $\mathcal{F}\diamond\mathcal{G}$ is the variety of the ideal $J\cap\mathbb{C}[\mathbf{x}]$. Therefore, considering $a_1>a_2>a_3>b_1>b_2>b_3>x>y>z$, a Gröbner basis w.r.t. the lex order provides the implicit equation of the cissoid.

In a second step, let us assume that we are given rational parametrizations of $\mathcal{F}$ and $\mathcal{G}$, say $\mathbf{f}=(f_1(u,v),f_2(u,v),f_3(u,v))$ and $\mathbf{g}=(g_1(s,t),g_2(s,t),g_3(s,t))$ respectively. Then, we want to compute the implicit equation of $\mathcal{F}\diamond\mathcal{G}$. One possibility is, of course, implicitize $\mathbf{f}$ and $\mathbf{g}$ and apply the method above. Alternatively, we consider the ideal

$$\begin{aligned} J &=<\text{numer}(\mathbf{x}-\mathbf{f}(u,v)-\mathbf{g}(s,t)),\mathbf{f}(u,v)\times\mathbf{g}(s,t),D(u,v,s,t)W-1> \\ &=<\text{numer}(x-f_1-g_1),\text{numer}(y-f_2-g_2),\text{numer}(z-f_3-g_3), \\ &\qquad f_2g_3-f_3g_2,f_3g_1-f_1g_3,f_1g_2-f_2g_1,D(u,v,s,t)W-1> \end{aligned} \tag{3}$$

in $\mathbb{C}[u, v, s, t, W, \mathbf{x}] = \mathbb{C}[u, v, s, t, W, x, y, z]$, where D is the least common multiple of all denominators in $\mathbf{f}(u, v)$ and $\mathbf{g}(s, t)$ and W is a new variable. Then $\mathcal{F} \diamond \mathcal{G}$ is the variety of the ideal $J \cap \mathbb{C}[\mathbf{x}]$. Therefore, considering $W > u > v > s > t > x > y > z$, a Gröbner basis w.r.t. the lex order provides the implicit equation of the cissoid.

In the last part of the section, we consider the problem of deriving a parametrization of the cissoid from parametrizations of $\mathcal{F}$ and $\mathcal{G}$. A direct method would be to compute the implicit equation of the cissoid, as above, to afterwards apply any surface parametrization algorithm. We consider here a different approach that, under certain hypotheses would provide directly a parametrization of the cissoid. This approach will be, indeed, the underling idea of later reasonings in this paper. Moreover, this direct approach will also work for non-rational parametrizations.

Let $\mathcal{E}$ be a real affine surface and $\mathbf{e}(u, v)$ a parametrization of $\mathcal{E}$. We say that $\mathcal{F}$, similarly for $\mathcal{G}$, is *parametrizable over* $\mathcal{E}$ if there exists a function $r(u, v)$ such that $r(u, v)\mathbf{e}(u, v)$. Note that, $\mathcal{F}$ is parametrizable over $\mathcal{E}$ if there exists a function $r(u, v)$ such that $\mathcal{F}$ is an r-augmentation of $\mathcal{E}$. In addition, we observe that if $\mathcal{E}$ is taken as the sphere of equation $x^2 + y^2 + z^2 = 1$, we get the notion of polar representation (see Def. 1 in Gruber and Peternell, 2013).

In this situation, we get the following proposition that shows how to parametrize the cissoid when one of the surface is parametrizable over the other.

Proposition 1. *Let $\mathcal{G}$ be parametrizable over $\mathcal{F}$ by $\mathbf{g}(u, v) = m(u, v)\mathbf{f}(u, v)$, where $\mathbf{f}(u, v)$ is a parametrization of $\mathcal{F}$. Then the cissoid $\mathcal{C} = \mathcal{F} \diamond \mathcal{G}$ (or a component of it) is represented by*

$$\mathbf{c}(u, v) = (1 + m(u, v))\mathbf{f}(u, v). \tag{4}$$

Remark 1. For potential applications, one may relate the cissoid construction with the homotopic deformation of the surfaces $\mathcal{F}$ and $\mathcal{G}$. More precisely, let us assume that $\mathcal{G}$ is rationally parametrizable over the rational surface $\mathcal{F}$. Let us say that $\mathbf{g}(u, v) := m(u, v)\mathbf{f}(u, v)$ where $\mathbf{f}$ rationally parametrizes $\mathcal{F}$, $m(u, v) \in \mathbb{R}(u, v)$, and $\mathbf{g}$ parametrizes $\mathcal{G}$. Then,

$$\mathcal{F} \diamond_{(\lambda, 1-\lambda, \mathbf{0})} \mathcal{G}$$

can be parametrized as

$$\lambda \mathbf{f}(u, v) + (1 - \lambda)\mathbf{g}(u, v).$$

Thus, if λ takes values in $[0, 1]$, the cissoid can be seen as a homotopy between the continuous functions (over their domains) $\mathbf{f}$ and $\mathbf{g}$ (see e.g. Danciger et al., 2009, page 208). This interpretation of the cissoid construction and, consequently, its analysis, could be of help in applications as, for instance, boundary evolution (see (Siddiqi, 2008)) or boundary deformations (see Fujimura and Kuo, 1999; Jaillet and Siméon, 2008).

3. Rationality of the cissoid surface

The cissoid construction is a generalization of the conchoid construction. In Sendra and Sendra (2008) the number of irreducible components is studied in case of the conchoid construction. Although in general the cissoid will be an irreducible variety, reducible components may appear. In the following, we use the notion 'rationality' when one component of the cissoid is rational. In addition, we recall that a rational parametrization $\mathbf{f}(u, v) \in \mathbb{C}(u, v)^3$, of a surface $\mathcal{F}$, is proper if the rational map $\mathbf{f} : \mathbb{C}^2 \to \mathcal{F} : (u, v) \mapsto \mathbf{f}(u, v)$ is invertible.

Let $\mathbf{f}(u, v)$ be a proper rational parametrization of $\mathcal{F}$. Taking the approach presented in Proposition 1, we try to parametrize $\mathcal{G}$ over $\mathcal{F}$. Thus, we look for a rational function $m(u, v)$ such that $m(u, v)\mathbf{f}(u, v)$ is a parametrization of $\mathcal{G}$. In order to do so we plug $w\mathbf{f}(u, v)$ into $G(x, y, z)$, where w is a new variable. Consequently,

$$H(w, u, v) := \text{numer}(G(w\mathbf{f}(u, v)) \in \mathbb{C}[w, u, v] \tag{5}$$

is a polynomial in w and u, v. The variety $\mathcal{H}$ defined by $H(w, u, v) = 0$ is called the *reparametrizing variety (associated to* $\mathbf{f}$*)*. Formally $\mathcal{H}$ depends on the chosen parametrization $\mathbf{f}(u, v)$ of $\mathcal{F}$. The following Lemma shows that the property that $\mathcal{G}$ is parametrizable over $\mathcal{F}$ is independent of the chosen parametrization $\mathbf{f}(u, v)$. Theorem 3 characterizes the property of being parametrizable over a surface by means of $\mathcal{H}$.

Lemma 2. *The property that $\mathcal{G}$ is parametrizable over $\mathcal{F}$ is independent of the chosen proper parametrization $\mathbf{f}(u, v)$ of $\mathcal{F}$.*

Proof. Let $\mathbf{f}(u, v)$ and $\bar{\mathbf{f}}(\bar{u}, \bar{v})$ be two different proper parametrizations of $\mathcal{F}$. Then there exists an invertible map $\psi(\bar{u}, \bar{v}) = (u, v)$ such that $\bar{\mathbf{f}}(\bar{u}, \bar{v}) = \mathbf{f}(\psi(\bar{u}, \bar{v}))$. Substituting (u, v) by $\psi(\bar{u}, \bar{v})$ in $\mathbf{g}(u, v) = m(u, v)\mathbf{f}(u, v)$ yields

$$(\mathbf{g} \circ \psi)(\bar{u}, \bar{v}) = (m \circ \psi)(\bar{u}, \bar{v})(\mathbf{f} \circ \psi)(\bar{u}, \bar{v}),$$
$$\bar{\mathbf{g}}(\bar{u}, \bar{v}) = \bar{m}(\bar{u}, \bar{v})\bar{\mathbf{f}}(\bar{u}, \bar{v}), \text{ with } (\bar{u}, \bar{v}) = \psi^{-1}(u, v).$$

Thus the property that $m(u, v)\mathbf{f}(u, v)$ parameterizes $\mathcal{G}$ is equivalent to that $\bar{m}(\bar{u}, \bar{v})\bar{\mathbf{f}}(\bar{u}, \bar{v})$ parameterizes $\mathcal{G}$. □

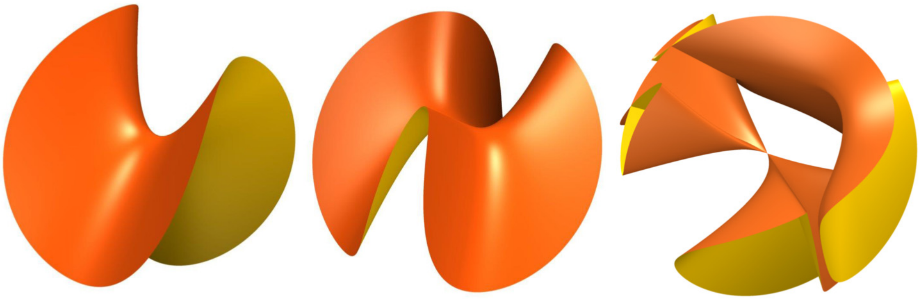

Fig. 1. $\mathcal{F}$ (left), $\mathcal{G}$ (center), $\mathcal{F} \diamond \mathcal{G}$ (right) in Example 5.

Theorem 3. *Let $\mathcal{D} \subset \mathbb{R}^3$ be the variety defined by the least common multiple of the denominators in $\mathbf{f}(u,v)$. The following statements are equivalent*

1. *$\mathcal{G}$ can be parametrized over $\mathcal{F}$ using $\mathbf{f}(u,v)$.*
2. *The reparametrizing variety $\mathcal{H}$ has a rational component, different from any irreducible component of $\mathcal{D}$, and this component admits a rational parametrization $\mathbf{q}(s,t) = (q_1, q_2, q_3)(s,t)$, such that the Jacobian of $\mathbf{g}(s,t) = q_1(s,t)\mathbf{f}(q_2(s,t), q_3(s,t))$ has rank 2.*

Furthermore, if (2) holds, then $\mathbf{g}(s,t)$ parametrizes $\mathcal{G}$.

Proof. Let $d(u,v)$ be a defining polynomial of $\mathcal{D}$, and let $H(w,u,v)$ be the defining polynomial of $\mathcal{H}$.

Let $\mathbf{g}(u,v) := m(u,v)\mathbf{f}(u,v)$ be a rational parametrization of $\mathcal{G}$, which implies $m \neq 0$. Then, $G(\mathbf{g}(u,v)) = 0$. Thus $H(m(u,v),u,v) = 0$, and hence $\mathbf{q}(u,v) = (q_1,q_2,q_3) := (m(u,v),u,v)$ parametrizes a component of $\mathcal{H}$, which is clearly different of $\mathcal{D}$. Moreover,

$$q_1(u,v)\mathbf{f}(q_2(u,v), q_3(u,v)) = m(u,v)\mathbf{f}(u,v),$$

whose Jacobian has rank 2.

Conversely, let $\mathbf{q}(s,t)$ be a rational parametrization of a component of $\mathcal{H}$, different of $\mathcal{D}$. Then $H(\mathbf{q}(s,t)) = 0$ and $d(q_2,q_3)(s,t) \neq 0$. H can be expressed as $H(w,u,v) = G(w\mathbf{f}(u,v))d(u,v)^n$ for some $n \in \mathbb{N}$. We consider the parametrization

$$\mathbf{g}(s,t) = q_1(s,t)\mathbf{f}(q_2(s,t), q_3(s,t)),$$

which satisfies $G(\mathbf{g}(s,t)) = 0$. Since the Jacobian of $\mathbf{g}(s,t)$ has rank two, $\mathbf{g}(s,t)$ is a rational parametrization of $\mathcal{G}$ over $\mathcal{F}$ using $\mathbf{f}$. □

Remark 2. We observe that the condition on the rank of the Jacobian in Theorem 3 cannot be avoided, as the following example shows. Consider the planes $\mathcal{F} : x = 0$ and $\mathcal{G} : y = 0$. Clearly, $\mathcal{G}$ cannot be parametrized over $\mathcal{F}$. However, if $\mathbf{f}(u,v) = (0,u,v)$, then $H(w,u,v) = u$ and $\mathcal{H}$ is rational. Nevertheless, any parametrization of $\mathcal{H}$ is of the form $(q_1(s,t), 0, q_3(s,t))$. But obviously the Jacobian of the parametrization $q_1\,\mathbf{f}(0,q_3) = (0,0,q_1q_3)$ has rank 1.

As a consequence of equation (4) and of Theorem 3, we have the following criterium for detecting rational cissoids.

Corollary 4. [Criterium of rationality] *Let $\mathbf{f}(u,v)$ be proper parametrization of $\mathcal{F}$ and the assumptions on $\mathcal{D}$ according to Theorem 3. If the reparametrizing variety $\mathcal{H}$ has a rational component $\mathbf{q}(s,t) = (q_1,q_2,q_3)(s,t)$ different from any irreducible component of $\mathcal{D}$ and if the Jacobian of $\mathbf{g} = q_1\mathbf{f}(q_2,q_3)$ has rank 2, then the cissoid $\mathcal{C} = \mathcal{F} \diamond \mathcal{G}$ has a component parametrized by*

$$\mathbf{c}(s,t) = \mathbf{f}(s,t) + \mathbf{g}(s,t) = (1 + q_1(s,t))\mathbf{f}(q_2(s,t), q_3(s,t)). \tag{6}$$

Example 5. Let $\mathcal{F}$ be the quadric defined by $F(x,y,z) = -xz + y + 1$ and $\mathcal{G}$ be the cubic surface defined by $G(x,y,z) = x^2y - z + 1$. $\mathcal{F}$ is a hyperbolic paraboloid, thus a ruled quadric, and $\mathcal{G}$ is a ruled cubic surface, equivalent to the *Whitney umbrella* $zy^2 = x^2$, up to a projective transformation ($(x,y,z) \mapsto (x/y, 1/y, z/y)$ composed with a translation). Using the ideal introduced in equation (2), the cissoid $\mathcal{F} \diamond \mathcal{G}$ is a 10-degree surface defined by (see Fig. 1)

$$\begin{aligned} &x^5y^2z^3 - 3x^4y^3z^2 - 3x^4y^2z^2 + 3x^3y^4z - 2x^3yz^4 + 6x^3y^3z + 2x^3yz^3 - x^2y^5 + 4x^2y^2z^3 \\ &\quad + 3x^3y^2z - 3x^2y^4 - 3x^2z^2y^2 - 3xy^3z^2 + xz^5 - 3x^2y^3 + 6x^2yz^2 + 3xy^3z - 2xz^4 \\ &\quad + zy^4 - z^4y - x^2y^2 - 3xy^2z + 2xyz^2 + xz^3 - y^4 + y^3z + yz^3 - z^4. \end{aligned}$$

$\mathcal{F}$ is parametrizable by $\mathbf{f}(u, v) = (u, uv - 1, v)$, and consequently the reparametrizing variety $\mathcal{H}$ is defined by

$$H(w, u, v) = w^3u^3v - w^3u^2 - wv + 1.$$

Since H is linear in v, it is obviously parametrized as

$$\mathbf{q}(u, w) = \left(w, u, \frac{u^2w^3 - 1}{w(u^3w^2 - 1)}\right) = (w, u, v(u, w)).$$

By Corollary 4, the cissoid surface $\mathcal{F} \diamond \mathcal{G}$ is rational and can be parametrized as

$$\begin{aligned}\mathbf{c}(u, w) &= (1 + q_1(u, w))\mathbf{f}(q_2(u, w), q_3(u, w)),\\ &= (1 + w)\,(u, uv(u, w) - 1, v(u, w))\,,\end{aligned}$$

and the Jacobian of $\mathbf{g}(u, w) = q_1(u, w)\mathbf{f}(q_2(u, w), q_3(u, w))$ has rank 2.

Corollary 6. *Let $\mathcal{F}$ be a rational surface and let $\mathcal{G}$ be a plane not passing through the origin. Then, the cissoid $\mathcal{F} \diamond \mathcal{G}$ is rational.*

Proof. Let $\mathbf{f}(u, v)$ be a proper parametrization of $\mathcal{F}$ and $G = a_1x + a_2y + a_3z + a_4 = \mathbf{a} \cdot \mathbf{x} + a_4$. $\mathbf{0} \notin \mathcal{G}$ implies $a_4 \neq 0$. To construct a rational parametrization $\mathbf{g}(u, v) = w(u, v)\mathbf{f}(u, v)$ of $\mathcal{G}$, the lines $w\mathbf{f}$ are intersected with $\mathcal{G}$. The unique intersection point leads to

$$w(u, v) = -\frac{a_4}{\mathbf{a} \cdot \mathbf{f}(u, v)}.$$

This agrees with the fact that the reparametrizing variety $\mathcal{H}$ is the numerator of $w(\mathbf{a} \cdot \mathbf{f}(u, v)) + a_4$ and, consequently, $\mathcal{H}$ is parametrized by $\mathbf{q}(u, v) = (-a_4/(\mathbf{a} \cdot \mathbf{f}), u, v)$. The plane $\mathcal{G}$ and the cissoid $\mathcal{C} = \mathcal{F} \diamond \mathcal{G}$ are parametrized by

$$\mathbf{g}(u, v) = \frac{-a_4}{\mathbf{a} \cdot \mathbf{f}(u, v)}\mathbf{f}(u, v), \text{ and } \mathbf{c}(u, v) = \mathbf{f}(u, v) + \mathbf{g}(u, v) = \frac{\mathbf{a} \cdot \mathbf{f}(u, v) - a_4}{\mathbf{a} \cdot \mathbf{f}(u, v)}\mathbf{f}(u, v),$$

respectively. □

4. The cissoid surface of an augmented ruled surface

Let us start this section by recalling the notion of an augmented ruled surface. We say that a rational surface is an augmented ruled surface, if it admits a parametrization of the form

$$T(u, v)\,(\mathbf{a}(u) + v\mathbf{b}(u))\,,$$

where T is a rational function in u, v.

We study the cissoid $\mathcal{C} = \mathcal{F} \diamond \mathcal{G}$ of two rational surfaces $\mathcal{F}$ and $\mathcal{G}$, where one, say $\mathcal{F}$, is an augmented ruled surface. At first we describe the general strategy for the analysis of the rationality of the cissoid. Later this is applied to some special cases.

Let $\mathcal{F}$ and $\mathcal{G}$ be represented by the respective parametrizations $\mathbf{f}(u, v) = T(u, v)\widetilde{\mathbf{f}}(u, v)$, where $\widetilde{\mathbf{f}}(u, v) = \mathbf{a}(u) + v\mathbf{b}(u)$, and $\mathbf{g}(s, t)$. Consider the family of planes

$$\alpha(u) : \mathbf{x} \cdot (\mathbf{a}(u) \times \mathbf{b}(u)) = 0$$

passing through the origin and the generating lines of the ruled surface $\widetilde{\mathcal{F}}$ defined by $\widetilde{\mathbf{f}}$. Intersecting $\mathcal{G}$ with the planes $\alpha(u)$ yields a family of curves $a(u) = \mathcal{G} \cap \alpha(u)$, see Fig. 2. Inserting the parametrization $\mathbf{g}(s, t)$ into $\alpha(u)$ results in an implicit representation of these curves $a(u)$

$$A(s, t; u) = \mathbf{g}(s, t) \cdot (\mathbf{a} \times \mathbf{b})(u) = 0. \tag{7}$$

Let us assume that $A(s, t; u) = 0$ defines a family of rational curves in the st-plane, with family parameter u. Then there exists a rational parametrization

$$\varphi(u, w) = (s(u, w), t(u, w)), \text{ with } A(s(u, w), t(u, w); u) = 0. \tag{8}$$

Substituting $\varphi(u, w)$ into $\mathbf{g}(s, t)$ yields $\mathbf{g}(s(u, w), t(u, w)) =: \mathbf{g}(u, w)$, such that the w-lines of $\mathbf{g}(u, w)$ are the planar curves $a(u) = \mathcal{G} \cap \alpha(u)$. Consequently, $\det(\mathbf{g}(u, w), \mathbf{a}(u), \mathbf{b}(u)) = 0$ holds.

The system of equations

$$\lambda\mathbf{g}(u, w) = \mathbf{a}(u) + v\mathbf{b}(u) = \widetilde{\mathbf{f}}(u, v) \tag{9}$$

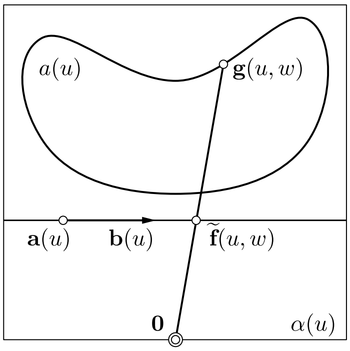

Fig. 2. Cissoid construction of augmented ruled surfaces.

is linear in λ and v, and has a rational solution $\lambda(u,w)$, $v(u,w)$. Substituting $v(u,w)$ in $\mathbf{f}(u,v)$ yields $\mathbf{f}(u,v(u,w)) =: \widetilde{\mathbf{f}}(u,w)$. This implies a rational parametrization $\widetilde{\mathbf{f}}(u,w) = T(u,w)\lambda(u,w)\mathbf{g}(u,w)$ of $\mathcal{F}$ with the property that $\widetilde{\mathbf{f}}(u,w)$ and $\mathbf{g}(u,w)$ are linearly dependent. Finally, the cissoid $\mathcal{C} = \mathcal{F} \diamond \mathcal{G}$ is rationally parametrized by

$$\begin{aligned}\mathbf{c}(u,w) &= \mathbf{f}(u,v(u,w)) + \mathbf{g}(u,w)\\ &= (1 + T(u,w)\lambda(u,w))\,\mathbf{g}(u,w).\end{aligned} \tag{10}$$

Theorem 7. *Let $\mathcal{F}$ and $\mathcal{G}$ be two rational surfaces, where $\mathcal{F}$ is an augmented ruled surface. The respective parametrizations are $\mathbf{f}(u,v) = T(u,v)(\mathbf{a}(u) + v\mathbf{b}(u))$ and $\mathbf{g}(s,t)$. If the generic planar intersection curve $a(u) = \alpha(u) \cap \mathcal{G}$, with a plane $\alpha(u) : \mathbf{x} \cdot (\mathbf{a}(u) \times \mathbf{b}(u)) = 0$, is rational, then the cissoid surface $\mathcal{C} = \mathcal{F} \diamond \mathcal{G}$ is rationally parametrized by (10).*

Example 8. We illustrate the method for a ruled surface $\mathcal{F}$ and a Steiner surface $\mathcal{G}$. The surfaces have the respective parametrizations

$$\mathbf{f}(u,v) = (u, v, u^2+1), \text{ and } \mathbf{g}(s,t) = (s, s^2+t, 1+t^2).$$

$\mathcal{F}$ is a parabolic cylinder with y-parallel lines and $\mathcal{G}$ is obtained by translating the parabolas $(s, s^2, 1)$ and $(0, t, t^2)$ along each other. The respective defining polynomials are

$$F = z - x^2 - 1, \text{ and } G = z - (y - x^2)^2 - 1.$$

Inserting $\mathbf{g}$ into the equation $\alpha(u) : (1+u^2)x - uz = 0$ of the planes through the lines of $\mathcal{F}$ yields $A(s,t;u) = -(1+u^2)s + u(1+t^2)$, which is simply solved by

$$s(t,u) = \frac{u(1+t^2)}{1+u^2}.$$

Substituting this expression in $\mathbf{g}(s,t)$, and solving the linear system $\lambda\mathbf{g}(t,u) = \mathbf{f}(u,v)$ gives

$$\begin{aligned}\lambda(t,u) &= \frac{u^2+1}{t^2+1},\\ v(t,u) &= \frac{tu^4 + 2tu^2 + t + u^2t^4 + 2u^2t^2 + u^2}{u^2t^2 + u^2 + t^2 + 1}.\end{aligned}$$

Finally, the cissoid $\mathcal{F} \diamond \mathcal{G}$, of implicit degree 12, is parametrized by

$$\begin{aligned}\mathbf{c}(t,u) &= \mathbf{f}(u, v(t,u)) + \mathbf{g}(t,u) = (1 + \lambda(t,u))\mathbf{g}(t,u)\\ &= \left(\frac{u(u^2+2+t^2)}{u^2+1}, \frac{(u^2+2+t^2)(tu^4 + 2tu^2 + t + u^2t^4 + 2u^2t^2 + u^2)}{(t^2+1)(u^2+1)^2}, u^2+2+t^2\right).\end{aligned}$$

According to Theorem 7, a sufficient condition for a rational construction is that the generic planar intersections of $\mathcal{G}$ are rational. It is known that there are only two possibilities, that $\mathcal{G}$ is either a ruled surface or a Steiner surface, see for instance Baker (1925), Vol 4, page 55. In addition, we observe that the reasoning above, can be analogously extended to the case where both surfaces $\mathcal{F}$ and $\mathcal{G}$ are augmented ruled surfaces, and to the case where $\mathcal{F}$ is an augmented ruled surface and $\mathcal{G}$ is an augmented Steiner surface. More precisely, we get the following corollary.

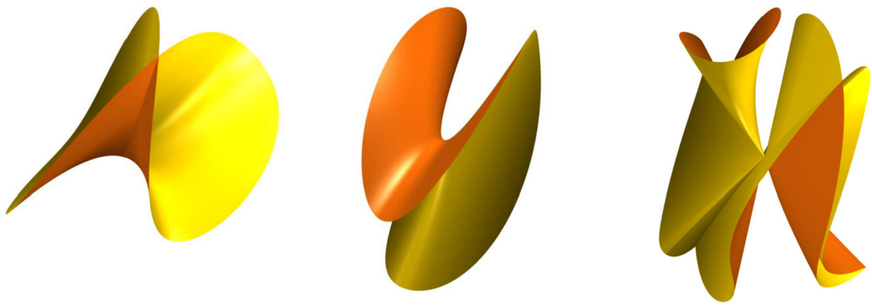

Fig. 3. $\mathcal{F}$ (left), $\mathcal{G}$ (center), $\mathcal{F} \diamond \mathcal{G}$ (right) in Example 10.

Corollary 9. *Let $\mathcal{F}$ be an augmented ruled surface. Then,*

1. *If $\mathcal{G}$ is an augmented ruled surface, then $\mathcal{F} \diamond \mathcal{G}$ is a rational surface.*
2. *If $\mathcal{G}$ is an augmented Steiner surface, then $\mathcal{F} \diamond \mathcal{G}$ is a rational surface.*
3. *If $\mathcal{G}$ is an irreducible quadric then $\mathcal{F} \diamond \mathcal{G}$ is a rational surface.*

We finish this section, with some illustrative examples.

Example 10. We consider the ruled surfaces $\mathcal{F}$ and $\mathcal{G}$ given parametrically by (see Fig. 3)

$$\mathbf{f}(u,v) = \mathbf{a}(u) + v\mathbf{b}(u) = (u+2, u^2+1, u^3+1) + v(1,1,u),$$
$$\mathbf{g}(s,t) = \mathbf{c}(s) + t\mathbf{d}(s) = (1, s+1, s^2+1) + t(1,0,s).$$

The respective implicit equations are

$$\begin{aligned} F &= -xy^2 + y^3 + 2\,yx - y^2 + yz - z^2 - x - 2\,y + z + 1 = 0, \\ G &= yx + y^2 - x - 3\,y - z + 3 = 0. \end{aligned}$$

The implicit equation of the curves $\alpha(u)$ is $A(s,t;u) = (\mathbf{c}(s) + t\mathbf{d}(s)) \cdot (\mathbf{a}(u) \times \mathbf{b}(u)) = 0$, from where one gets that

$$t(s,u) = -\frac{\det(\mathbf{a},\mathbf{b},\mathbf{c})(s,u)}{\det(\mathbf{a},\mathbf{b},\mathbf{d})(s,u)},$$

and hence

$$t(u,s) = -\frac{s^2u^2 - su^3 - s^2u + su^2 - u^3 - s^2 + 2\,su + 2\,u^2 - s - 1}{su^2 - su - s - u + 1}.$$

Solving the linear system corresponding to $\lambda(u,s)\mathbf{g}(s,u) = \mathbf{f}(u, v(u,s))$ yields

$$\begin{aligned} \lambda(u,s) &= \frac{su^2 - su - s - u + 1}{2\,s^2 - su - u + 1}, \\ v(u,s) &= \frac{-s^2u^2 + su^3 - s^2u + su^2 + u^3 - 3s^2 - su - u^2}{2s^2 - su - u + 1}. \end{aligned}$$

Finally, we compute the parametrization of the cissoid,

$$(1 + \lambda(u,s))\,\mathcal{Q}(u,s),$$

where $\mathcal{Q}(u,s)$ is

$$\left(-\frac{s^2u^2 - su^3 - s^2u - u^3 - s^2 + 3\,su + 2\,u^2 + u - 2}{su^2 - su - s - u + 1}, \frac{su^2 + 2s^2 - 2su - s - 2u + 2}{2s^2 - su - u + 1}, \right.$$
$$\left. \frac{s^2u^3 - s^2u^2 + su^3 - 3s^2u - su^2 + 2s^2 - su - u + 1}{su^2 - su - s - u + 1}\right).$$

Example 11. Consider an augmented ruled surface $\mathcal{F}$ and a Steiner surface $\mathcal{G}$. Similarly to Example 8, let

$$\mathbf{f}(u,v) = \frac{v}{1+u^2}(u, v, u^2+1), \text{ and } \mathbf{g}(s,t) = (s, s^2+t, 1+t^2),$$

where $\mathbf{f} = T\widetilde{\mathbf{f}}$ with $T = v/(1+u^2)$ and $\widetilde{\mathbf{f}} = (u, v, u^2+1)$. $\mathcal{F}$ is of degree four and the zero set of $F = 4x^4 + x^2z^2 - 4x^2y - yz^2 + y^2$. The family of planes

$$\alpha(u) : (1+u^2)x = uz$$

intersects $\mathcal{F}$ in a pair of degree two curves, passing through $(0,0,0)$. Substituting $\mathbf{g}(s,t)$ into $\alpha(u)$ gives a family of conics. Analogously to Example 8 we obtain

$$s(t,u) = \frac{u(1+t^2)}{1+u^2}.$$

The system of equations $\lambda\mathbf{g}(t,u) = f(u,v)$ is solved by $\lambda = v = 0$ and the non-trivial solution

$$\lambda(t,u) = \frac{t^4u^2 + 2t^2u^2 + t(u^2+1)^2 + u^2}{(u^2+1)(t^2+1)^2},$$
$$v(t,u) = \frac{tu^4 + 2tu^2 + t + u^2t^4 + 2u^2t^2 + u^2}{(u^2+1)(t^2+1)}.$$

Denoting the analogous solutions in Example 8 by $\widetilde{v}(t,u)$ and $\widetilde{\lambda}(t,u)$, we have $v = \widetilde{v}$ and $\lambda = T(u, v(t,u))\widetilde{\lambda}$. The coordinate representation of the final cissoid $\mathcal{F} \diamond \mathcal{G}$, parametrized by $\mathbf{c}(t,u) = (1+\lambda(t,u))\mathbf{g}(t,u)$ is rather lengthy and therefore omitted here.

Example 12. We illustrate the cissoid construction for a ruled surface and a quadric. Consider the ruled surface $\mathcal{F} : x^3 - xyz - x^2 + y^2 = 0$ that is parametrized by

$$\mathbf{f}(u,v) = (vu+1, u^2v+u, v+u).$$

Let the ellipsoid $\mathcal{G} : 4x^2 + 4y^2 + z^2 - 4 = 0$ be parametrized by

$$\mathbf{g}(s,t) = \left(2\,\frac{s}{s^2+t^2+1}, 2\,\frac{t}{s^2+t^2+1}, 2\,\frac{s^2+t^2-1}{s^2+t^2+1}\right).$$

We get $A(s,t;u) = -su + t = 0$ and thus $t = us$. The linear system $\lambda\mathbf{g}(u,s) = \mathbf{f}(u,v)$ results in

$$\lambda(u,s) = \frac{(u^2-1)(u^2s^2+s^2+1)}{2\left(su^2+u+s\right)(us-1)}.$$

Finally, we get that a parametrization of $\mathcal{F} \diamond \mathcal{G}$ is given by $\mathbf{p}(u,s) = \left(\frac{p_1}{p}, \frac{p_2}{p}, \frac{p_3}{p}\right)$, where

$$\begin{aligned}
p_1(u,s) &= \left(u^4s^2 + 2u^3s^2 + 2us^2 + u^2 - s^2 - 2u - 2s - 1\right)s \\
p_2(u,s) &= \left(u^4s^2 + 2u^3s^2 + 2us^2 + u^2 - s^2 - 2u - 2s - 1\right)us \\
p_3(u,s) &= \left(u^4s^2 + 2u^3s^2 + 2us^2 + u^2 - s^2 - 2u - 2s - 1\right)\left(u^2s^2 + s^2 - 1\right) \\
p(u,s) &= \left(u^2s + u + s\right)(us-1)\left(u^2s^2 + s^2 + 1\right)
\end{aligned}$$

We observe that in this case the cissoid has degree 8 and its defining polynomial is

$$\begin{aligned}
C = {} & 4x^8 + z^2x^4 - 2z^2x^5 + z^2x^6 - 8x^7 + 8x^5yz + 8x^4yz - 2x^4yz^3 + 2yx^3z^3 \\
& -8yzx^6 + x^2y^2z^4 - 6y^2z^2x^2 + 4y^2x^6 + 4y^2z^2x^4 + 2x^3y^2z^2 - 4y^2x^4 \\
& -2xy^3z^3 - 8x^4y^3z + 4y^4z^2x^2 - 4y^4x^2 + 8x^3y^4 + y^4z^2 - 8xy^5z + 4y^6.
\end{aligned}$$

Acknowledgements

This work has been partially funded by Ministerio de Economía y Competitividad, and by the European Regional Development Fund (ERDF), under the Project MTM2014-54141-P. Last two authors belongs to the Research Group ASYNACS (Ref. CCEE2011/R34). Part of this work was developed while M. Peternell was visiting J.R. Sendra at the Universidad de Alcalá in the frame of the project Giner de los Rios.

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