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# Geometric Properties of Bisector Surfaces

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This paper studies algebraic and geometric properties of curve–curve, curve–surface, and surface–surface bisectors. The computation is in general difficult since the bisector is determined by solving a system of nonlinear equations. Geometric considerations will help us to determine several distinguished curve and surface pairs which possess elementary computable bisectors. Emphasis is on low-degree rational curves and surfaces, since they are of particular interest in surface modeling. © 2000 Academic Press

## 1. INTRODUCTION

Given two geometric objects $O_1$ and $O_2$ in Euclidean 3-space $E^3$, their *bisector* $B$ is defined as the locus of equidistant points from $O_1$ and $O_2$. The distance is measured orthogonal to both objects. Thus, bisector $B$ is the set of centers of spheres touching both $O_1$ and $O_2$. We do not require that the distance from $B$ to $O_1$ and $O_2$ be minimal and we only discuss the *untrimmed bisector*; see [12]. The objects shall be points, smooth curves, and surfaces. If the objects $O_1$, $O_2$ are just continuous and possess edges, the bisector $B$ can be computed applying the offset operation discussed in Section 3.3. We will not discuss this in detail.

If $O = O_1 = O_2$ is a solid object, $B$ shall be called an untrimmed *self-bisector*. The *medial axis* or *skeleton*—see [1, 36]—is a subset of $B$. It consists of centers of maximal inscribable spheres.

The computation of the bisector and medial axis is in general difficult. We want to determine several families of curves and surfaces which admit an elementary bisector computation. These constructions shall be called *basic algorithms*. Bisectors of "complicated" objects can be approximated by approximating the given objects by those objects which admit elementary bisector constructions. In particular we will discuss rational curve and surface pairs which possess rational bisector surfaces. Several results derived here may also be found in [8–12, 16].



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The article is organized as follows. Section 2 collects some facts about planar bisectors. Section 3 introduces point–surface and point–curve bisectors in space. Section 4 discusses algebraic properties of point–object bisectors. We will study a quadratic transformation which maps an object onto the point–object bisector. Circles, spheres, planes, and envelopes of them are distinguished with respect to this mapping. Section 5 studies curve–curve bisectors in space. Sections 6 and 7 present special examples of point–surface and sphere–surface bisectors. We deal with curve–surface bisectors in Section 8, and Section 9 briefly discusses surface–surface bisectors. Section 10 presents results about bisectors of developable surfaces. Furthermore, we will point out relations to geometric optics.

A possible method for studying bisectors from a geometric point of view is the geometry of oriented spheres and planes in $E^3$, called Laguerre geometry. A detailed description of the planar case is given in [26]. Some facts and details can also be found in [32]. A general monograph of sphere geometry is [4]. Here, we mainly use constructive and analytic geometry.

Points in $E^3$ are denoted by their coordinate vectors $\mathbf{x} = (x_1, x_2, x_3)$; points in Euclidean plane $E^2$ are denoted analogously. The scalar product and vector product of two vectors $\mathbf{x}$, $\mathbf{y}$ are denoted by $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$, respectively. Section 4 uses a different notation for points and planes in projective extension $P^3$ of $E^3$, which is explained there. When computing point–curve or point–surface bisectors we will always assume that a coordinate system has been chosen such that this point is the origin $O = (0, 0, 0)$ (or $O = (0, 0)$ in the planar case).

## 2. BISECTOR CONSTRUCTIONS IN THE PLANE

The bisector of two planar curves has been discussed in detail in the literature; see for instance [12, 13, 26, 34] and references therein. We want to collect some geometric properties which possess analogs in the spatial case.

### 2.1. Point–Curve Bisector in the Plane

DEFINITION 2.1. The bisector $B$ of two planar curves $C_1$, $C_2$ is the locus of points being equidistant to both curves. Distances are measured orthogonal to the input curves $C_1$, $C_2$.

First of all, the bisector of two points is the line of symmetry. The bisector of point $O$ and not incident line $L$ is a parabola with focal point $O$, axis orthogonal to $L$, and directrix $L$.

Let $C$ be a planar curve parametrized by $\mathbf{c}(t) = (c_1(t), c_2(t))$ and let $B$ be the bisector with respect to the origin $O = (0, 0)$. By definition, $B$ contains all centers of circles which are tangent to $C$ and which pass through $O$. Moreover, the lines of symmetry

$$\sigma(t) : \mathbf{c}(t) \cdot \left( \mathbf{x} - \frac{1}{2}\mathbf{c}(t) \right) = 0 \tag{1}$$

of $O$ and $\mathbf{c}(t)$ are tangent lines of $B$. Thus, we refer to (1) as dual representation of $B$. The conversion to a representation of $B$ as a point set is obtained by intersecting lines $\sigma(t)$ and $\dot{\sigma}(t)$. Therefore, points $\mathbf{b}(t)$ of the bisector $B$ are solutions of the linear system

$$\sigma(t) : 2\mathbf{c}(t) \cdot \mathbf{x} = \mathbf{c}(t) \cdot \mathbf{c}(t), \qquad \dot{\sigma}(t) : \dot{\mathbf{c}}(t) \cdot \mathbf{x} = \mathbf{c}(t) \cdot \dot{\mathbf{c}}(t). \tag{2}$$

On the other hand, $\mathbf{b}(t)$ is the intersection point of the symmetry line $\sigma(t)$ and the curve normal $\mathbf{c}(t) + \lambda \mathbf{n}(t)$, where $\mathbf{n} = (-\dot{c}_2, \dot{c}_1)$ is a normal vector of $C$. This construction leads

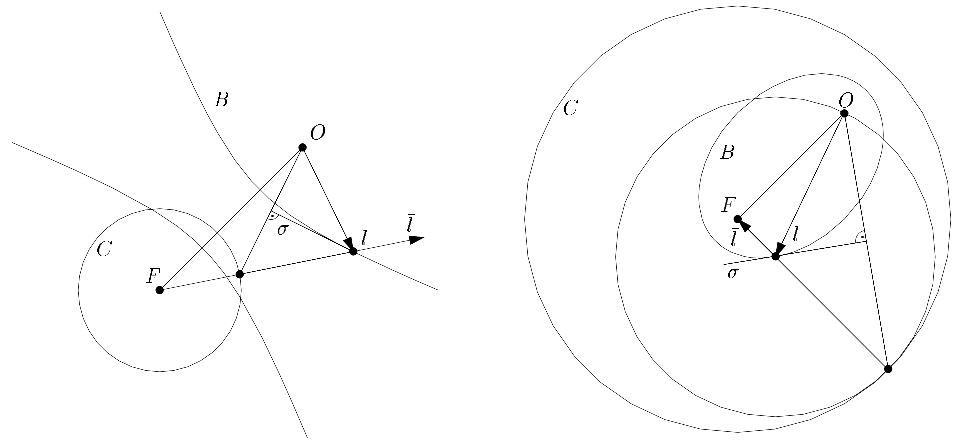


**FIG. 1.** Point–curve bisector and optical interpretation.

to the parametrization

$$\mathbf{b}(t) = \mathbf{c}(t) - \frac{1}{2}\frac{\mathbf{c}(t)\cdot\mathbf{c}(t)}{\mathbf{c}(t)\cdot\mathbf{n}(t)}\mathbf{n}(t), \tag{3}$$

which solves (2). Figure 1 shows elementary examples. The bisector between point $O$ and circle $C$ is an ellipse or hyperbola depending on whether $O$ is inside or outside of $C$.

The point–curve bisector is a basic algorithm and serves to compute curve–curve bisectors; see [12]. The reason is that the bisector $B$ of curves $C_1$ and $C_2$ is the envelope of a family of curves $B_1(t)$. This family consists of bisectors of $C_1$ and a variable point $\mathbf{c}_2(t) \in C_2$.

### 2.2. Curve–Curve Bisectors and Geometric Optics

The relation between bisector and geometric optics follows immediately from its definition. Light rays $l$ radiating from a point source $O = C_1$ are reflected at the bisector $B$ in such a way that the reflected light rays $\bar{l}$ are orthogonal to the given curve $C = C_2$; see Fig. 1. This principle holds in general if $C_1$ is an arbitrary planar curve.

*Summary 2.1.* Light rays $l$ perpendicular to $C_1$ will be reflected at the bisector $B$ such that the reflected rays $\bar{l}$ are perpendicular to $C_2$.

The curve $C_2$ is said to be an *anticaustic* of illumination orthogonal to $C_1$ and reflection at $B$. The definition is symmetric with respect to $C_1$ and $C_2$. Thus, bisectors are mirror curves with respect to the given illumination.

A further property of bisectors is their invariance under offsetting the input curves. Let $C_1$ be an oriented circle with radius $d$, centered at $O$ and with outward-oriented normals, and let $C_2$ be an arbitrary planar curve. The bisector of $C_1$, $C_2$ equals the bisector of $O$ and $D$, with $D$ as an offset curve to $C_2$ at distance $-d$. The orientation is important since we have to substitute $C_1$ and $C_2$ by their offsets $O$ and $D$ at distance $-d$. This principle holds in general for the bisector $B$ of two oriented curves $C_1$, $C_2$.

*Summary 2.2.* Substituting $C_1$ and $C_2$ by offset curves at oriented distance $d$ leaves the bisector $B$ unchanged.

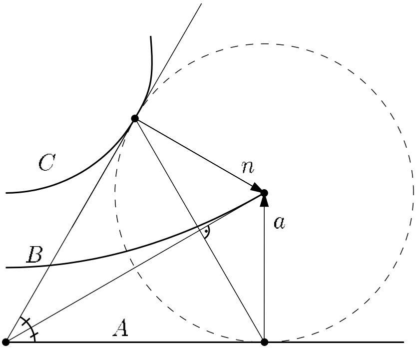


**FIG. 2.** Line–curve bisector.

Considering rational planar curves $C_1, C_2$, their bisector $B$ will not be rational in general. A special case implying rational bisectors is the following. Let $C_2 = A$ be a straight line given by equation $\mathbf{a} \cdot \mathbf{x} = 0$ with $\|\mathbf{a}\| = 1$ (A passes through $O$). Let $C_1 = C$ be a rational curve with parametrization $\mathbf{c}(t)$ possessing rational unit normal vectors

$$\mathbf{n}(t) = \frac{1}{\|\dot{\mathbf{c}}\|}(-\dot{c}_2, \dot{c}_1)(t).$$

Curve $C$ is called a *rational offset curve* or *rational PH-curve*; see [30]. Depending on the orientation of $A$ and $C$ (see Fig. 2), the bisector $B$ is for instance parametrized by

$$\mathbf{b}(t) = \mathbf{c}(t) - \frac{\mathbf{a} \cdot \mathbf{c}(t)}{(\mathbf{a} - \mathbf{n}(t)) \cdot \mathbf{n}(t)} \mathbf{n}(t).$$

The tangent lines of $B$ are lines of symmetry of $A$ and the tangents of $C$; see Fig. 2. We consider an illumination $l$ orthogonal to $A$. Light rays $l$ are reflected at the mirror $B$ in a way such that the reflected rays $\bar{l}$ are orthogonal to $C$. The curve $C$ is called anticaustic for parallel illumination and reflection at $B$. In terms of geometric optics this says that *mirror curves B to rational PH-curves C for parallel light rays are rational*. Similar statements will hold in the spatial case.

*Summary 2.3.* The bisector of two circles and the bisector of a line and a rational PH-curve is always rational.

Of course, there are many more pairs of rational curves possessing rational bisectors. But in general it is not easy to decide whether the bisector of two rational input curves is rational.

*Remark.* $E^2$ shall be embedded into $E^3$ with coordinates $x, y, r$ as plane $r = 0$. Consider oriented planar curves $C_1, C_2$ in $E^2$. We will give a spatial interpretation of the bisector construction. Let $\Gamma_1, \Gamma_2$ be developable surfaces through $C_1, C_2$ which possess constant slope $\pi/4$ with respect to $E^2$. This implies that the inclination angle of generating lines of $\Gamma_1, \Gamma_2$ with $E^2$ equals $\pi/4$. Each developable surface $\Gamma_i$ is the graph of the signed distance function to curve $C_i$, $i = 1, 2$. Further, let $D$ be the intersection curve $\Gamma_1 \cap \Gamma_2$. Thus, the orthogonal projection of $D$ onto $E^2$ is the bisector $B$ of $C_1, C_2$. If $D$ is parametrized by $\mathbf{d}(t) = (d_1, d_2, d_3)(t)$, the bisector $B$ possesses the parametrization $\mathbf{b}(t) = (d_1, d_2)(t)$. The radii of circles centered at $\mathbf{b}(t)$ which are tangent to $C_1, C_2$ equal $d_3(t)$.

This method is studied and applied in [8, 26, 32]. We will see later that this method works analogously in the spatial case.

If $C_1 = C_2 = C$, we call $B$ a self-bisector of $C$. $B$ is again the orthogonal projection of the self-intersection $D$ of the developable of constant slope $\Gamma$ through $C$. The "true" (or trimmed) bisector is a subset of $B$. The trimming procedure can be realized by applying a visibility algorithm to the intersection $D$ with respect to $\Gamma$; see [32]. A similar idea for trimmed bisector construction is presented in [3].

## 3. BISECTOR CONSTRUCTIONS IN SPACE

DEFINITION 3.1. The bisector $B$ of two surfaces $F_1$, $F_2$ is the locus of points being equidistant to both surfaces. Distances are measured orthogonal to the input surfaces $F_1$, $F_2$.

Let us collect some elementary properties. The bisector of two points $\mathbf{p}$ and $\mathbf{q}$ is their *plane of symmetry*

$$\sigma : (\mathbf{p} - \mathbf{q}) \cdot \left( \mathbf{x} - \frac{1}{2}(\mathbf{p} + \mathbf{q}) \right) = 0.$$

The bisector surface of a point $\mathbf{p}$ and a line $G$ is a *parabolic cylinder*. Its generator lines are perpendicular to the plane, spanned by $\mathbf{p}$ and $G$; see Fig. 3. The bisector surface of a point $\mathbf{p}$ and a plane $E$ is a *paraboloid of revolution* with focal point $\mathbf{p}$ and axis perpendicular to $E$.

Next, the bisector surface $B$ of a smooth surface $F$ and origin $O$ shall be computed. By definition, the bisector $B$ contains the centers of all spheres tangent to $F$ and passing through $O$. Let $F$ be parametrized by $\mathbf{f}(u, v)$. The symmetry planes of $O$ and $\mathbf{f}(u, v)$ are

$$\sigma(u, v) : \mathbf{f}(u, v) \cdot \left( \mathbf{x} - \frac{1}{2}\mathbf{f}(u, v) \right) = 0. \tag{4}$$

Bisector $B$ is the envelope of planes $\sigma(u, v)$. Thus, we call (4) a dual representation (parametrization) of $B$. The conversion to a point representation of $B$ is done by intersecting planes $\sigma(u, v)$ with partial derivative planes $\sigma_u(u, v)$ and $\sigma_v(u, v)$. Thus, a parametrization $\mathbf{b}(u, v)$ of $B$ solves the linear system

$$\sigma(u, v) : \mathbf{f} \cdot \mathbf{x} = \frac{1}{2}\mathbf{f} \cdot \mathbf{f}, \qquad \sigma_u(u, v) : \mathbf{f}_u \cdot \mathbf{x} = \mathbf{f} \cdot \mathbf{f}_u, \qquad \sigma_v(u, v) : \mathbf{f}_v \cdot \mathbf{x} = \mathbf{f} \cdot \mathbf{f}_v. \tag{5}$$

Parametrization $\mathbf{b}(u, v)$ is also obtained by intersecting symmetry planes $\sigma(u, v)$ with the

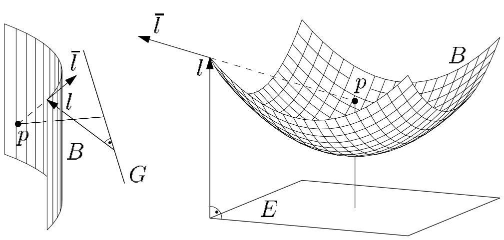


**FIG. 3.** Point–line and point–plane bisector.

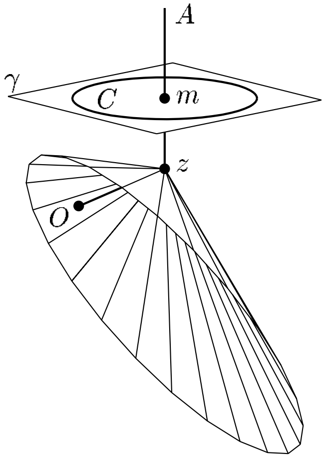


**FIG. 4.** Point–circle bisector in space.

surface normals $\mathbf{f} + \lambda \mathbf{f}_u \times \mathbf{f}_v$. Thus, a solution of (5) is given by

$$\mathbf{b}(u, v) = \mathbf{f} - \frac{1}{2} \frac{\mathbf{f} \cdot \mathbf{f}}{\det(\mathbf{f}, \mathbf{f}_u, \mathbf{f}_v)} (\mathbf{f}_u \times \mathbf{f}_v). \tag{6}$$

### 3.1. Point–Curve Bisector in Space

The bisector surface of a curve $C$ and a point $O$ shall be computed (Fig. 4). We will again assume $O$ to be the origin (0, 0, 0). Curve $C$ shall be parametrized by $\mathbf{c}(t)$. The bisector $B$ is enveloped by the planes of symmetry of $O$ and $\mathbf{c}(t)$. This family is

$$\sigma(t) : \mathbf{c}(t) \cdot \left( \mathbf{x} - \frac{1}{2}\mathbf{c}(t) \right) = 0. \tag{7}$$

Thus, $B$ is always a *developable surface*. This answers the question of Elber and Kim [9], of which (space) curves possess a developable point–curve bisector surface.

A developable surface is a special kind of ruled surface, so it carries generating lines $g(t)$. All regular points of a fixed generator $g(t_0)$ possess the constant tangent plane $\sigma(t_0)$. The generating lines $g(t)$ can be computed as intersection lines of $\sigma(t)$ and

$$\dot\sigma(t) : \dot{\mathbf{c}}(t) \cdot (\mathbf{x} - \mathbf{c}(t)) = 0.$$

The singular point on $g(t_0)$ is obtained by intersecting $g(t_0)$ with $\ddot\sigma(t_0)$. The notation of a parametrization of $B$ as a point set is compact if we use Plücker coordinates (as “black box”) for the generating lines $g(t)$. We form the vectors

$$\mathbf{l} := \mathbf{c} \times \dot{\mathbf{c}}, \qquad \bar{\mathbf{l}} := (\mathbf{c} \cdot \dot{\mathbf{c}})\mathbf{c} - \frac{1}{2}(\mathbf{c} \cdot \mathbf{c})\dot{\mathbf{c}},$$

where $\mathbf{l}$ is a direction vector of $g$ and $\mathbf{l} \cdot \bar{\mathbf{l}} = 0$ holds. The foot point of $g$ with respect to the origin $O$ is given by

$$\mathbf{f} := \frac{1}{(\mathbf{l} \cdot \mathbf{l})} \mathbf{l} \times \bar{\mathbf{l}}. \tag{8}$$

This yields a parametrization of $B$ in the form $\mathbf{b}(s, t) = \mathbf{f}(t) + s\mathbf{l}(t)$.

EXAMPLE. Consider a circle $C$ with center $\mathbf{m}$ lying in a plane $\gamma$ (Fig. 4). $C$ is parametrized by $\mathbf{c}(t)=\mathbf{m}+\mathbf{c}_1(t)$ with

$$\mathbf{c}_1=\mathbf{a}\cos t+\mathbf{b}\sin t,\qquad \text{with } \|\mathbf{a}\|=\|\mathbf{b}\| \text{ and } \mathbf{a}\cdot\mathbf{b}=0.$$

The radius $r_C$ of $C$ equals $\|\mathbf{c}_1\|$, and $\mathbf{c}_1\cdot\dot{\mathbf{c}}_1=0$ holds. Further, $\mathbf{c}_1\times\dot{\mathbf{c}}_1=\mathbf{n}$ is a constant normal vector of $\gamma$. The symmetry plane $\sigma$ and its first and second derivatives are

$$\sigma(t):(\mathbf{c}_1+\mathbf{m})\cdot\mathbf{x}=\frac{1}{2}\left(\mathbf{m}^2+r_C^2\right)+\mathbf{m}\cdot\mathbf{c}_1,\qquad \dot\sigma(t):\dot{\mathbf{c}}_1\cdot\mathbf{x}=\mathbf{m}\cdot\dot{\mathbf{c}}_1,\qquad \ddot\sigma(t):\mathbf{c}_1\cdot\mathbf{x}=\mathbf{c}_1\cdot\mathbf{m}.$$

The bisector $B$ of $C$ and $O$ is a quadratic cone or cylinder. In the case of a cone, its vertex $\mathbf{z}$ is obtained by intersecting $\sigma\cap\dot\sigma\cap\ddot\sigma$,

$$\mathbf{z}=\mathbf{m}+\frac{r_C^2-\mathbf{m}^2}{2\mathbf{m}\cdot\mathbf{n}}\mathbf{n}. \tag{9}$$

If $\mathbf{m}\cdot\mathbf{n}\neq 0$, $B$ is a cone and its vertex $\mathbf{z}$ lies always on the axis $A$ of the circle $C$. The bisector $B$ is parametrized by

$$\mathbf{b}(t,s)=\mathbf{z}+s(\mathbf{n}+\mathbf{m}\times\dot{\mathbf{c}}_1(t)).$$

The axis $A$ of $C$ and the line connecting $\mathbf{z}$ and $O$ are real *focal lines* of the cone $B$.

$B$ is a cylinder, if $\mathbf{m}\cdot\mathbf{n}=0$, which expresses incidence of $O$ and $\gamma$. Thus, this is a planar problem and $B$ possesses the planar bisector of $O$ and $C$ as its cross section. The generating lines of $B$ are parallel to $A$. Its focal lines pass through the focal points of the cross section $B\cap\gamma$ and are $A$ and the normal to $\gamma$ through $O$.

### 3.2. Point–Curve Bisector of Curves on Surfaces

It is important to discuss how point–curve bisectors are related to point–surface bisectors in the case of curves on surfaces. Let a surface $F$ be given by a parametrization $\mathbf{f}(u,v)$. Any curve $C$ on $F$ can be represented by $\mathbf{c}(t)=\mathbf{f}(u(t),v(t))$. We know already that the bisector $B_F$ of $F$ and the origin $O$ is enveloped by planes $\sigma(u,v):\mathbf{f}(u,v)\cdot(\mathbf{x}-\frac{1}{2}\mathbf{f}(u,v))=0$, whereas the bisector $B_C$ of $C$ and $O$ is enveloped by the one-parameter family $\sigma(t)$: $\mathbf{c}(t)\cdot(\mathbf{x}-\frac{1}{2}\mathbf{c}(t))=0$. Obviously, each plane $\sigma(t)$ occurs also as plane $\sigma(u,v)$, which means that $B_C$ *is a developable surface tangent to* $B_F$. Two examples shall illustrate this.

#### 3.2.1. *Planar Curves*

We can assume that the plane $F$ is defined by the equation $z=1$. Let $C\subset F$ be a planar curve parametrized by $\mathbf{c}(t)=(c_1(t),c_2(t),1)$. The bisector $B_F$ of $F$ and $O$ is a paraboloid of revolution, and the bisector $B_C$ is a developable surface, tangent to $B_F$ in a curve $D$. It is an elementary property of $B_F$ that $D$ is the orthogonal projection of $C$ onto $B_F$. Thus, $D$ is parametrized by

$$\mathbf{d}(t)=\left(c_1,c_2,\frac{1}{2}\left(1-c_1^2-c_2^2\right)\right).$$

Since the generating lines of $B_C$ possess direction vectors $\mathbf{c}\times\dot{\mathbf{c}}$, a parametrization of $B_C$ is

$$\mathbf{b}(t,s)=\left(c_1,c_2,\frac{1}{2}\left(1-c_1^2-c_2^2\right)\right)+s(-\dot c_2,\dot c_1,c_1\dot c_2-c_2\dot c_1).$$

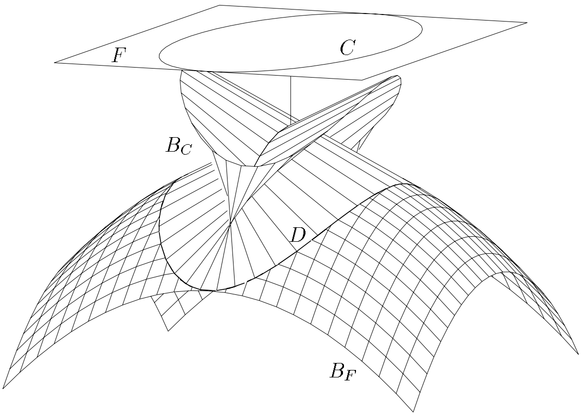


**FIG. 5.** Bisector of planar curve and point in space.

Figure 5 illustrates the bisector surface $B_C$ of $O$ and an ellipse $C \subset F$. The paraboloid $B_F$ possesses $O$ as focal point (not visible).

### 3.2.2. *Spherical Curves*

Let $F$ be a sphere with center $\mathbf{m}$ and radius $r$ and let $C$ be a curve on $F$. The bisector $B_F$ with respect to $O$ is a quadric of revolution with focal points in $O$ and $\mathbf{m}$. The bisector $B_C$ is a developable surface, tangent to $B_F$ along a curve $D$. The curve of contact $D$ is obtained by intersecting the lines $\mathbf{m} + \lambda(\mathbf{c}(t) - \mathbf{m})$ with the symmetry planes $\sigma(t)$ of $\mathbf{c}(t)$ and $O$. This leads to

$$\mathbf{d}(t) = \mathbf{m} + \frac{\mathbf{c} \cdot \left(\frac{1}{2}\mathbf{c} - \mathbf{m}\right)}{\mathbf{c} \cdot (\mathbf{c} - \mathbf{m})}(\mathbf{c} - \mathbf{m}).$$

Figure 6 shows the bisector of a spherical algebraic curve $C$ of order 4. Further, one could

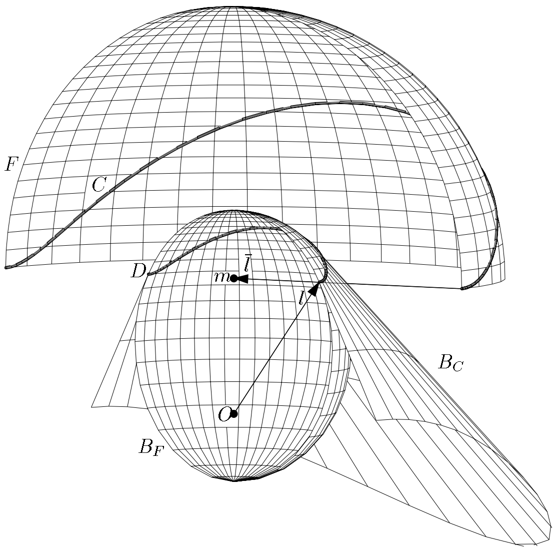


**FIG. 6.** Bisector of spherical curve and point in space.

study bisectors of curves on cones and cylinders of revolution, or Dupin cyclides. These families of surfaces possess similar properties than planes and spheres; see Sections 6.1 and 6.2. This shall not be investigated here.

### 3.3. Surface–Surface Bisectors and Geometric Optics

Analogously to Section 2.2, the construction of bisectors is related to geometric optics. These are obvious generalizations of the planar case.

*Summary 3.1.* Light rays $l$ perpendicular to $F_1$ are reflected at the bisector $B$ in a way such that the reflected rays $\overline{l}$ are perpendicular to $F_2$.

$F_2$ is called *anticaustic* for illumination orthogonal to $F_1$ and reflection at $B$. Thus, construction of bisectors is equivalent to the construction of mirror surfaces.

Let $F_1$ be an oriented sphere with radius $d$, center $O$, and outward-oriented normals. The bisector of $F_1$, $F_2$ equals the bisector of $O$ and $D$, with $D$ as offset surface to $F_2$ at distance $-d$. Orientation is important since we have to substitute $F_1$ and $F_2$ both by their offsets $O$ and $D$ at distance $-d$. More generally formulated we obtain

*Summary 3.2.* Substituting $F_1$ and $F_2$ by offset surfaces at oriented distance $d$ leaves the bisector $B$ unchanged.

Considering rational surfaces $F_1$, $F_2$, their bisector $B$ will in general not be rational. But if one surface, say $F_2$ is a plane and $F_1$ is a rational surface, which possesses rational unit normals, their bisector surface is rational. A parametrization $\mathbf{b}(t)$ of the bisector is given in Section 9.1.

*Remark.* Analogously to the planar case we embed the spatial problem into 4–space with coordinates $x, y, z, r$. Consider two oriented surfaces $F_1$, $F_2$ in the hyperplane $E^3 : r = 0$. Let $\Gamma_1, \Gamma_2$ be two hypersurfaces of constant slope $\pi/4$ with respect to $E^3$, which pass through $F_1$, $F_2$. Further, let $B$ be the orthogonal projection of $D = \Gamma_1 \cap \Gamma_2$ onto $E^3$. Since $\Gamma_1, \Gamma_2$ form the same angle with $E^3$, the surface $B$ is the bisector of $F_1$ and $F_2$. Assume that $\mathbf{d}(u, v) = (d_1, d_2, d_3, d_4)(u, v)$ is a parametrization of $D$; the projection $B$ onto $E^3$ is parametrized by $\mathbf{b}(u, v) = (d_1, d_2, d_3)(u, v)$. The radius function of spheres centered at $\mathbf{b}(u, v)$ and tangent to surfaces $F_1$, $F_2$ is $d_4(u, v)$. This method is applied in [8, 32].

If $F = F_1 = F_2$, the untrimmed self-bisector can be computed as an orthogonal projection of the self-intersection of the hypersurface $\Gamma$ onto $E^3$. The trimmed bisector is obtained by applying a visibility algorithm; see [32].

## 4. GEOMETRIC BACKGROUND AND ALGEBRAIC PROPERTIES

For studying algebraic properties of point–curve and point–surface bisectors we will use projective extensions $P^2$ and $P^3$ of Euclidean spaces $E^2$ and $E^3$, respectively. $E^3$ is extended to $P^3$ by adding points at infinity. We will describe just the spatial case since the planar situation is completely analogous. Each point at infinity is defined by a bundle of parallel lines. All these points at infinity form the plane at infinity.

A mathematical description of $P^3$ is obtained by using homogeneous coordinates $\mathsf{x} = (x_0, x_1, x_2, x_3)$. They are only determined up to a scalar multiple, which says that

$$\mathsf{x} \quad \text{and} \quad \mathsf{x}\rho,\ \rho \in \mathbb{R}\backslash\{0\}$$

define the same point. It is convenient to denote points by $\mathsf{x}\mathbb{R}$. A plane in $P^3$ is given by a linear homogeneous equation

$$u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0.$$

The vector $\mathsf{u} = (u_0, u_1, u_2, u_3)$ is called a homogeneous coordinate vector of that plane. Since coordinates $u_i$ are also just determined up to a scalar multiple, planes shall be denoted by $\mathbb{R}\mathsf{u}$. The left and right scalar multiples in plane and point coordinates are just a convention to distinguish them.

Points at infinity are characterized by $x_0 = 0$. Points not at infinity ($x_0 \neq 0$) possess Cartesian coordinates $(x, y, z)$. The conversion from homogeneous to Cartesian coordinates is done by

$$(x_0, x_1, x_2, x_3) \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) = (x, y, z).$$

An analogous description works in projective plane $P^2$. Points and lines are represented by $\mathsf{x}\mathbb{R} = (x_0, x_1, x_2)\mathbb{R}$ and $\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, u_1, u_2)$, respectively.

### 4.1. Algebraic Properties of Planar Point–Curve Bisectors

It shall be shown that the point–curve bisector is obtained by applying a quadratic mapping to the curve. This will lead to further geometric properties in particular for algebraic curves. We will define the following quadratic *point-to-line mapping*,

$$\begin{aligned} \beta \colon \mathsf{x}\mathbb{R} &\mapsto \mathbb{R}\mathsf{u}, \\ (x_0, x_1, x_2) &\mapsto \left(-\frac{1}{2}(x_1^2 + x_2^2), x_0x_1, x_0x_2\right) = (u_0, u_1, u_2). \end{aligned} \tag{10}$$

This maps points $\mathsf{x}\mathbb{R} = (x_0, x_1, x_2)\mathbb{R}$ to symmetry lines $\mathbb{R}\mathsf{u}$ with respect to the origin $O = (1, 0, 0)\mathbb{R}$. If $\mathsf{x}\mathbb{R} = \mathsf{c}\mathbb{R}$ is a point on the curve $C$, the symmetry line $\mathbb{R}\mathsf{u}$ envelops the bisector $B$ of $C$ and $O$. Thus, $\beta$ will be called the *bisector map*. To apply this to bisectors of algebraic curves, we also need the inverse mapping

$$\begin{aligned} \beta^{-1} \colon \mathbb{R}\mathsf{u} &\mapsto \mathsf{x}\mathbb{R}, \\ (u_0, u_1, u_2) &\mapsto \left(-\frac{1}{2}(u_1^2 + u_2^2), u_0u_1, u_0u_2\right) = (x_0, x_1, x_2). \end{aligned} \tag{11}$$

Let $C$ be an irreducible planar algebraic curve in $P^2$ given by the polynomial equation

$$C(x_0, x_1, x_2) \colon a_n(x_1, x_2) + \cdots + a_{n-j}(x_1, x_2)x_0^j + \cdots + a_0x_0^n = 0, \tag{12}$$

where $x_i$ are unknowns and $a_j(x_1, x_2)$ are homogeneous polynomials of degree $j$. Inserting the right-hand side of (11) into Eq. (12) gives an equation of the bisector $B$ in line coordinates,

$$\begin{aligned} B(u_0, u_1, u_2) \colon u_0^n a_n(u_1, u_2) + \cdots + \left(\frac{-1}{2}\right)^j \left(u_1^2 + u_2^2\right)^j a_{n-j}(u_1, u_2) u_0^{n-j} \\ + \cdots + a_0 \left(\frac{-1}{2}\right)^n \left(u_1^2 + u_2^2\right)^n = 0. \end{aligned} \tag{12a}$$

Notation *equation in line coordinates* shall express that $B(t_0, t_1, t_2) = 0$ holds for homogeneous coordinates of tangent lines $T = \mathbb{R}(t_0, t_1, t_2)$ of the bisector $B$.

To study the algebraic properties of the bisector $B$ we recall that the *order* of an algebraic curve is the algebraically counted number of intersection points with a line in $P^2$. The *class* of an algebraic curve is the algebraically counted number of tangent lines of the curve that pass through a fixed point.

Since $\beta$ is a quadratic point to line mapping, curves $C$ of degree $d$ are in general mapped to curves $\beta(C) = B$ of class $2d$. This rule possesses several exceptions. At first, Eq. (12a) says that if $O = (1, 0, 0)\mathbb{R}$ is $k$-fold point of $C$, that means $a_0 = \cdots = a_{k-1} = 0$; Eq. (12a) is divisible by $u_0^k$. Thus, the class of $B$ reduces by $k$.

If $a_n(x_1, x_2)$ is divisible by $x_1^2 + x_2^2$, Eq. (12a) is divisible by $(u_1^2 + u_2^2)$. Thus, the class of the bisector reduces by 2. This actually occurs if $C$ is a circle. The fact that $x_1^2 + x_2^2$ is a factor of $a_n$ has a geometric interpretation. This says that the curve $C$ contains the conjugate complex points $\mathsf{i}, \bar{\mathsf{i}}$ at infinity. These are called *ideal points*, and their homogeneous coordinates are

$$\mathsf{i} = (0, 1, i) \qquad \text{and} \qquad \bar{\mathsf{i}} = (0, 1, -i).$$

*Summary 4.1.* If an algebraic curve $C$ of order $d$ possesses $k$-fold points at the ideal points the class of bisector $B$ reduces by $2k$.

### 4.2. Algebraic Properties of Point–Surface Bisectors

Analogously to the planar case we will discuss algebraic properties of the point–surface bisector with help of bisector mapping. To obtain a more compact notation we will use the following abbreviation for homogeneous point and plane coordinates:

$$\mathsf{x}\mathbb{R} = (x_0, x_1, x_2, x_3)\mathbb{R} = (x_0, \mathbf{x})\mathbb{R}, \qquad \text{with } \mathbf{x} = (x_1, x_2, x_3), \tag{13}$$

$$\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, u_1, u_2, u_3) = \mathbb{R}(u_0, \mathbf{u}), \qquad \text{with } \mathbf{u} = (u_1, u_2, u_3). \tag{14}$$

Similar to the planar case we define the bisector mapping

$$\begin{aligned} \beta\colon \mathsf{x}\mathbb{R} &\mapsto \mathbb{R}\mathsf{u}, \\ (x_0, \mathbf{x})\mathbb{R} &\mapsto \mathbb{R}\left(-\frac{1}{2}\mathbf{x}\cdot\mathbf{x}, x_0\mathbf{x}\right) = \mathbb{R}(u_0, \mathbf{u}), \end{aligned} \tag{15}$$

which maps points $\mathsf{x}\mathbb{R}$ to symmetry planes $\mathbb{R}\mathsf{u}$ with respect to the origin $O = (1, 0, 0, 0)\mathbb{R}$. The inverse mapping of (15) is

$$\begin{aligned} \beta^{-1}\colon \mathbb{R}\mathsf{u} &\mapsto \mathsf{x}\mathbb{R}, \\ \mathbb{R}(u_0, \mathbf{u}) &\mapsto \left(-\frac{1}{2}\mathbf{u}\cdot\mathbf{u}, u_0\mathbf{u}\right)\mathbb{R} = (x_0, \mathbf{x})\mathbb{R}. \end{aligned} \tag{16}$$

Let $F$ be an irreducible algebraic surface in $P^3$ given by the polynomial equation

$$F(x_0, \mathbf{x})\colon a_n(\mathbf{x}) + \cdots + a_{n-j}(\mathbf{x})x_0^j + \cdots + a_0 x_0^n = 0,$$

where $x_i$ are unknowns and $a_j(\mathbf{x})$ are homogeneous polynomials of degree $j$ in $x_1, x_2, x_3$. The equation of the bisector $B$ follows by inserting expressions of $\beta^{-1}$ into $F$. Thus, we obtain

$$B(u_0, \mathbf{u}) : u_0^n a_n(\mathbf{u}) + \cdots + \left(\frac{-1}{2}\right)^j (\mathbf{u}\cdot\mathbf{u})^j a_{n-j}(\mathbf{u}) u_0^{n-j} + \cdots + a_0 \left(\frac{-1}{2}\right)^n (\mathbf{u}\cdot\mathbf{u})^n = 0. \quad (17)$$

The *order* of an algebraic surface is the algebraically counted number of intersection points with a line in $P^3$. The *class* of an algebraic surface is the algebraically counted number of tangent planes of the surface passing through a line.

Since $\beta$ is a quadratic point to plane mapping, surfaces of degree $d$ are in general mapped to surfaces of class $2d$. A point $\mathsf{b}\mathbb{R}$ is a *base point* of the transformation $\beta$ if $\beta(\mathsf{b}) = (0, 0, 0, 0)$ which does not represent points in $P^3$. From (15) it follows that the origin $O = (1, 0, 0, 0)$ is a base point. If $F$ has a $k$-fold point in $O$, Eq. (17) is divisible by $u_0^k$ and the class of the bisector $B$ reduces by $k$. Further, the *absolute conic*

$$J : x_0 = 0, x_1^2 + x_2^2 + x_3^2 = 0 \quad (18)$$

consists of base points. If $a_n(\mathbf{x})$ is divisible by $\mathbf{x}\cdot\mathbf{x}$, surface $F$ contains the absolute conic $J$. Formula (17) tells us that $B$ is divisible by $\mathbf{u}\cdot\mathbf{u}$.

*Summary 4.2.* Let $F$ be an algebraic surface of order $n$ which contains $J$ with multiplicity $k$. Then, the bisector $B$ is of class $2(n-k)$.

Examples for such surfaces are spheres, tori, Dupin cyclides and Darboux cyclides. The point–sphere bisectors are quadrics. Since cyclides of order 4 possess $J$ as a double curve, their bisectors possess the same order; see Section 6.2.

*Remark.* Studying the above representations (10) and (15), we see that $\beta$ can be decomposed in the following way,

$$\mathsf{x}\mathbb{R} \mapsto \kappa(\mathbf{x}\mathbb{R}) \mapsto \pi(\kappa(\mathbf{x}\mathbb{R})) \mapsto \sigma(\pi(\kappa(\mathsf{x}\mathbf{R}))) = \beta(\mathsf{x}\mathbb{R}),$$
$$(x_0, \mathbf{x})\mathbb{R} \mapsto ((\mathbf{x}\cdot\mathbf{x}), x_0\mathbf{x})\mathbb{R} \mapsto \mathbb{R}(-(\mathbf{x}\cdot\mathbf{x}), x_0\mathbf{x}) \mapsto \mathbb{R}(-\tfrac{1}{2}(\mathbf{x}\cdot\mathbf{x}), x_0\mathbf{x}),$$

where $\kappa$ and $\pi$ are *inversion* and *polarity* with respect to the unit sphere $S^2$ and $\sigma$ is a scaling with factor $\frac{1}{2}$. $S^2$ can be replaced by an arbitrary sphere centered at $O$. A point $\mathsf{x}\mathbb{R} = (x_0, \mathbf{x})$ is mapped at first onto the point $\kappa(\mathsf{x}\mathbb{R})$. Then, polarity $\pi$ maps $\kappa(\mathsf{x}\mathbb{R})$ to its polar plane $\pi(\kappa(\mathbf{x}\mathbb{R}))$ with respect to $S^2$. The final scaling is a similarity with center $O$ and factor $\frac{1}{2}$; see Fig. 7. In short form we write

$$\beta = \sigma \circ \pi \circ \kappa. \quad (19)$$

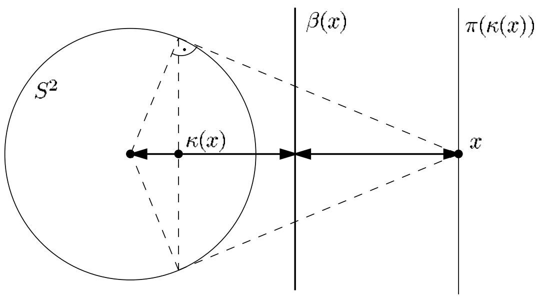


**FIG. 7.** Decomposition of the bisector mapping.

Scaling $\sigma$ and polarity $\pi$ are bijective mappings. Thus, the base points of $\beta$ are exactly base points of the inversion $\kappa$.

Similarity $\sigma$ scales the distance of a plane from the origin. This is done by scaling just the first coordinate $u_0$. In the case of $u_1^2 + u_2^2 + u_3^2 = 1$, which are normalized plane coordinates, the first coordinate $u_0$ equals the oriented distance of the plane $\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, \mathbf{u})$ from $O$.

*Remark.* Up to the scaling by $\frac{1}{2}$, the bisector mapping $\beta$ equals the inverse *pedal transformation*. The pedal transformation with respect to the origin $O$ maps planes $\mathbb{R}\mathsf{u}$ not passing through $O$ to its foot points $\mathsf{f}\mathbb{R}$ with respect to $O$. In homogeneous coordinates, the pedal transformation reads as

$$\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, \mathbf{u}) \mapsto (-\mathbf{u} \cdot \mathbf{u}, u_0\mathbf{u})\mathbb{R} = \mathsf{f}\mathbb{R}. \tag{20}$$

The above decomposition also works for parametric surfaces. We will now return to Cartesian coordinates. Let the surface $F$ be parametrized by $\mathbf{f}(u, v)$. The inversion $\kappa$ maps the surface point $\mathbf{f}$ onto

$$\mathbf{g}(u, v) = \kappa(\mathbf{f}(u, v)) = \frac{1}{\mathbf{f}(u, v) \cdot \mathbf{f}(u, v)}\mathbf{f}(u, v).$$

Polarity $\pi$ maps $\mathbf{g}$ to the polar plane $\tau$ (with respect to the unit sphere), whose equation is

$$\pi(\mathbf{g}(u, v)) = \tau(u, v) : -\mathbf{f}(u, v) \cdot \mathbf{f}(u, v) + \mathbf{f}(u, v) \cdot \mathbf{x} = 0.$$

We apply the scaling $\sigma$ and calculate the envelope of planes $\sigma(\tau(u, v))$. This exactly leads to the linear system (5) and the parametrization (6) of the bisector surface.

### 4.3. Algebraic Properties of a Point–Curve Bisector in Space

The above decomposition (19) also serves to compute the bisector of the point $O$ and a curve $C$. In particular, let $C$ be an algebraic curve of order $d$. The image curve $D = \kappa(C)$ is of order $2d$, if $C$ contains no base points. If $C$ contains $O$ or intersects the absolute conic $J$, the order of $D$ reduces by the multiplicity of $O$ and intersection multiplicity of $J \cap C$. Since the polarity $\pi$ is a point-to-plane mapping, $\pi(D)$ is the envelope of a one-parameter family of planes and thus a developable surface. The class of $\pi(D)$ is $2d$ in general.

Algebraic curves $C$ are given by polynomial equations and $C$ is considered an intersection of algebraic surfaces

$$C = F_1 \cap F_2.$$

At most, two equations are sufficient but note that for instance the twisted cubic is a complete intersection *not* of two surfaces but of three quadrics.

The bisector surface $B$ of $O$ and $C$ can be constructed via bisector surfaces $B_i$ of $F_i$ and $O$. Since $C$ is the intersection of surfaces $F_i$, the bisector $B$ is the intersection of $B_i$. We have to note that surfaces $B_i$ are considered to be two-parameter families of tangent planes in this context. Thus, intersection $B_1 \cap B_2$ is a one-parameter family of tangent planes which envelope an algebraic developable surface.

For instance, remember the point–circle bisector discussed in Section 3.1. Let $\gamma$ be the plane containing $C$. Applying $\beta = \sigma \circ \pi \circ \kappa$ we will obtain at first a circle $\kappa(C)$ in a plane $\alpha$. The polarity $\pi$ maps points of $\kappa(C)$ to a family of planes passing through the pole $\mathbf{z} = \pi^{-1}(\alpha)$

of $\alpha$. Since $\pi$ (and also $\pi^{-1}$) is a linear mapping, these planes envelope a quadratic cone; see Fig. 4.

If $O \in \gamma$, inversion $\kappa$ maps $\gamma$ onto itself. It follows that $\mathbf{z}$ is at infinity. Polarity $\pi$ maps $\kappa(C)$ onto a quadratic cylinder with generator lines perpendicular to $\gamma$.

## 5. CURVE–CURVE BISECTOR IN 3-SPACE

A detailed study of bisector surfaces of two space curves was done by Elber and Kim [9]. They proved that two rational space curves $C$ and $D$ possess a rational bisector surface $B$. The construction is as follows. Let $\mathbf{c}(u)$ and $\mathbf{d}(v)$ be rational parametrizations of space curves $C$ and $D$. A sphere $S$ is tangent to $C$ at a point $\mathbf{c}$ if its center is located in the normal plane to $C$ at $\mathbf{c}$. Thus, the center of a sphere tangent to $C$ and $D$ is located in the normal planes of $C$, $D$. Additionally, this center has to be contained in the symmetry plane $\sigma$ of points $\mathbf{c}$ and $\mathbf{d}$. A parametrization $\mathbf{b}(u, v)$ of $B$ is obtained as solution of the linear system

$$
\begin{aligned}
N_C : \dot{\mathbf{c}} \cdot (\mathbf{x} - \mathbf{c}) &= 0, && \text{normal plane to } C\\
N_D : \dot{\mathbf{d}} \cdot (\mathbf{x} - \mathbf{d}) &= 0, && \text{normal plane to } D\\
\sigma : (\mathbf{d} - \mathbf{c}) \cdot \left(\mathbf{x} - \frac{1}{2}(\mathbf{c} + \mathbf{d})\right) &= 0, && \text{symmetry plane of } C \text{ and } D.
\end{aligned}
\tag{21}
$$

We note that $\dot{\mathbf{c}}(u)$ is the derivative with respect to $u$, but $\dot{\mathbf{d}}(v)$ is the derivative with respect to $v$. We want to derive a formula for the parametrization $\mathbf{b}(u, v)$. The line of intersection $G = N_C \cap N_D$ can be represented by the vectors (compare Section 3.1)

$$
\mathbf{g} = \dot{\mathbf{c}} \times \dot{\mathbf{d}}, \qquad \overline{\mathbf{g}} = (\mathbf{d} \cdot \dot{\mathbf{d}})\dot{\mathbf{c}} - (\mathbf{c} \cdot \dot{\mathbf{c}})\mathbf{d}.
$$

Intersecting $G(u, v)$ with the plane of symmetry $\sigma(u, v)$ leads to the following parametrization of $B$,

$$
\mathbf{b}(u, v) = \frac{1}{(\mathbf{d} - \mathbf{c}) \cdot (\dot{\mathbf{c}} \times \dot{\mathbf{d}})} \left(-\frac{1}{2}(\mathbf{c}^2 - \mathbf{d}^2)(\dot{\mathbf{c}} \times \dot{\mathbf{d}}) + (\mathbf{d} - \mathbf{c}) \times ((\mathbf{d} \cdot \dot{\mathbf{d}})\dot{\mathbf{c}} - (\mathbf{c} \cdot \dot{\mathbf{c}})\dot{\mathbf{d}})\right). \tag{22}
$$

Points $\mathbf{p}_i = (u_i, v_i)$ for which $G(u_i, v_i) \subset \sigma(u_i, v_i)$ are called base points of the parametrization. The numerator and denominator of (22) vanish at $(u_i, v_i)$.

EXAMPLE: It is known that the bisector surface of two skew lines $G_1, G_2$ in space is a hyperbolic paraboloid $B$. Its axis is the common normal of $G_1, G_2$. Formula (22) does not represent the straight lines on $B$ but the parabolas; see Fig. 8. This follows from the

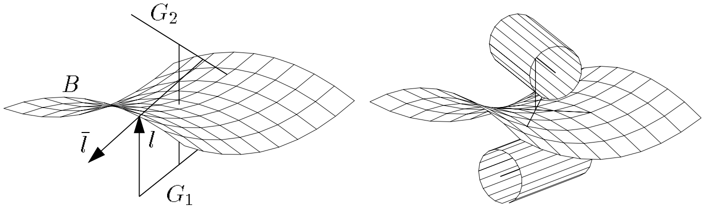


**FIG. 8.** Bisector of two lines and two cylinders of same radii.

geometric generation. We fix one point $\mathbf{y}_2$ on $G_2$. The bisector of $G_1$ and $\mathbf{y}_2$ is a parabolic cylinder $B_2$. The intersection of $B_2$ with a normal plane to $G_2$ through $\mathbf{y}_2$ gives a parameter curve of $B$.

Assume that a Cartesian coordinate system has been chosen such that the $z$-axis is the common normal of $G_1$ and $G_2$. Let these lines be parametrized by

$$G_1 = (u\cos\phi, u\sin\phi, d), \qquad \text{and} \qquad G_2 = (v\cos\phi, -v\sin\phi, -d),$$

where $\phi$ denotes the angle between $G_1$ and the $x$-axis. An elementary calculation shows that $B$ is parametrized by

$$\mathbf{b}(u, v) = (2d\sin\phi(u+v), 2d\cos\phi(u-v), -2\cos\phi\sin\phi(u+v)(u-v)).$$

It is biquadratic and possesses base points at infinity defined by the lines $u+v=0$ and $u-v=0$. A bilinear parametrization in $s, t$ can be obtained by the substitution $s=u+v$, $t=u-v$.

Figure 8 shows that the bisector $B$ of two cylinders $F_1$, $F_2$ of revolution with equal radii $r=r_1=r_2$ is also a hyperbolic paraboloid. Using the invariance under offsetting, $B$ can be computed as bisector of their offsets at distance $-r$. These offset surfaces degenerate to their axes of revolution. We will see later in Section 10 that the bisector surface for two cylinders (or cones) of revolution is of fourth order, in general.

The point–curve bisectors are completely described in Section 3.1. From a computational viewpoint, the point–circle bisector is of importance, since it is simple enough to serve as a basic algorithm. This means that the computation of point–curve bisectors of complicated curves can be approximated by point–circle bisector computations. There are algorithms for approximating a given curve by a sequence of circular arcs; see [14, 17, 20, 24, 27, 35]. Thus, the developable bisector surface is approximated by a sequence of quadratic cones which are the bisectors of the fixed point and the sequence of circular arcs.

In the following we discuss some low-degree examples for curve–curve bisectors. They can serve as basic algorithms for $C^1$-approximation of bisector surfaces of arbitrary space curves.

### 5.1. Circle–Line Bisector

One can immediately insert parametric representations $\mathbf{c}(u)$ and $\mathbf{g}(v)$ of a circle $C$ and a line $G$ into formula (21) and obtain a parametrization of the bisector $B$. We want to discuss this geometrically.

We will fix a point $\mathbf{c}(u_0)$ on the circle $C$ and calculate the bisector $B_G(u_0)$ of $\mathbf{c}(u_0)$ and $G$. We know already that $B_G(u_0)$ is a parabolic cylinder. Exactly those points of $B_G(u_0)$ which lie in the normal plane $N_C(u_0)$ to $C$ through $\mathbf{c}(u_0)$ are contained in the bisector $B$. Thus, $B$ is parametrized by a one-parameter family of parabolas

$$\mathbf{b}(u_0, v) = B_G(u_0) \cap N_C(u_0) \qquad \text{for } u_0 \in \mathbb{R}.$$

All planes $N_C(u_0)$ lie in a pencil of planes with the rotational axis of $C$ as a common line. Further, along each parabola $\mathbf{b}(u_0, v)$ the bisector $B$ possesses $B_G(u_0)$ as a contact developable surface. The parabolas possess two common points on the axis of the pencil.

On the other hand, we fix a point $\mathbf{g}(v_0)$ on $G$ and compute the bisector $B_C(v_0)$ of $C$ and $\mathbf{g}(v_0)$. From earlier statements it is clear that $B_C(v_0)$ is a quadratic cone with vertex on the axis of $C$. The intersection of $B_C(v_0)$ with the normal plane $N_G(v_0)$ to $G$ through $\mathbf{g}(v_0)$ is also a conic. So we obtain a second family of conics on $B$,

$$\mathbf{b}(u, v_0) = B_C(v_0) \cap N_G(v_0) \qquad \text{for } v_0 \in \mathbb{R}.$$

Along each conic $\mathbf{b}(u, v_0)$ the bisector surface $B$ is in contact with $B_C(v_0)$. The second family of conics also lies in a pencil of planes, orthogonal to $G$.

Such surfaces have been studied in detail by Degen [5, 6]. At that time and earlier these surfaces were called *double Blutel conic surfaces*. Nowadays the notation *supercyclides* is more convenient. These are algebraic surfaces of order $\leq 4$. Some subfamilies of them can be obtained as projective images of Dupin cyclides. For more information see [7, 23, 33] and the references therein.

### 5.2. Circle–Circle Bisector

Let the circles $C$ and $D$ be parametrized by $\mathbf{c}(u)$ and $\mathbf{d}(v)$ (Fig. 9). These curves lie in planes $\gamma$ and $\delta$, respectively. We proceed as in Section 5.1. We fix a point $\mathbf{c}(u_0)$ on $C$ and compute the bisector surface $B_D(u_0)$ of $\mathbf{c}(u_0)$ and $D$. We know that $B_D(u_0)$ is a quadratic cone with vertex on the axis $d$ of the circle $D$; see Section 3.1. So, the parameter curve $\mathbf{b}(u_0, v)$ of the bisector $B$ of $C$ and $D$ is the planar intersection of $B_D(u_0)$ with the normal plane $N_C(u_0)$ through $\mathbf{c}(u_0)$. Since these normal planes $N_C$ all pass through the axis $A$ of the circle $C$, this first family of parameter conics lies in planes through $A$. Further, the tangent planes along a fixed parameter conic $\mathbf{b}(u_0, v)$ envelop the quadratic cone $B_D(u_0)$.

We exchange $C$ with $D$ and compute parameter curves $\mathbf{b}(u, v_0)$. So it is clear that the bisector surface $B$ carries two families of conics, lying in planes through the axes of the given circles $C$, $D$. Further, along each conic there exists a quadratic tangent cone with

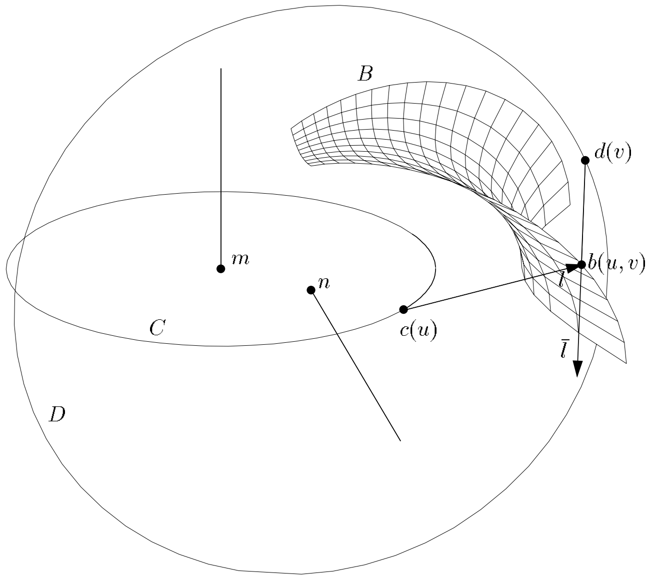


**FIG. 9.** Circle–circle bisector in space.

vertex on an axis of the circles. Thus, bisector $B$ is again a supercyclide. We omit the discussion of all types of supercyclides occurring here.

The practical calculation works as follows. We set

$$\begin{aligned}\mathbf{c}(u) &= \mathbf{m} + \mathbf{p}\cos u + \mathbf{q}\sin u = \mathbf{m} + \mathbf{c}_1(u),\\ \mathbf{d}(v) &= \mathbf{n} + \mathbf{r}\cos v + \mathbf{s}\sin v = \mathbf{n} + \mathbf{d}_1(v),\end{aligned} \tag{23}$$

where $\mathbf{m}$ and $\mathbf{n}$ are the centers of $C$ and $D$, and we require that

$$\|\mathbf{p}\| = \|\mathbf{q}\| = r_C, \qquad \mathbf{p}\cdot\mathbf{q} = 0, \qquad \|\mathbf{r}\| = \|\mathbf{s}\| = r_D, \qquad \text{and} \qquad \mathbf{r}\cdot\mathbf{s} = 0.$$

The length of $\mathbf{p}$ and $\mathbf{q}$ equals the radius $r_C$ of $C$, and additionally $\mathbf{p}$ and $\mathbf{q}$ are orthogonal. Analogously for $\mathbf{r}$ and $\mathbf{s}$. Further it is clear that $\mathbf{c}_1\cdot\dot{\mathbf{c}}_1 = 0$ and $\mathbf{d}_1\cdot\dot{\mathbf{d}}_1 = 0$ hold. The vectors $\mathbf{c}_1$, $\mathbf{d}_1$ and their derivative vectors of $\dot{\mathbf{c}}_1$, $\dot{\mathbf{d}}_1$ satisfy

$$\|\mathbf{c}_1\| = \|\dot{\mathbf{c}}_1\| = r_C \qquad \text{and} \qquad \|\mathbf{d}_1\| = \|\dot{\mathbf{d}}_1\| = r_D.$$

The cross products

$$\mathbf{c}_1\times\dot{\mathbf{c}}_1 = \mathbf{A}, \qquad \mathbf{d}_1\times\dot{\mathbf{d}}_1 = \mathbf{B}$$

are vectors of constant length. $\mathbf{A}$ and $\mathbf{B}$ are perpendicular to planes $\gamma$ and $\delta$, respectively, containing circles $C$ and $D$. Using these notations we insert (23) into formula (22). We will study denominator and vector part (numerator) separately. At first the denominator reads as

$$\begin{aligned} b_0 &= (\mathbf{d}-\mathbf{c})\cdot(\dot{\mathbf{c}}\times\dot{\mathbf{d}})\\ &= (\mathbf{n}+\mathbf{d}_1-\mathbf{m}-\mathbf{c}_1)\cdot(\mathbf{c}_1\times\mathbf{d}_1)\\ &= (\mathbf{n}-\mathbf{m})\cdot(\dot{\mathbf{c}}_1\times\dot{\mathbf{d}}_1) + \mathbf{d}_1\cdot(\dot{\mathbf{c}}_1\times\dot{\mathbf{d}}_1) - \mathbf{c}_1\cdot(\dot{\mathbf{c}}_1\times\dot{\mathbf{d}}_1)\\ &= (\mathbf{n}-\mathbf{m})\cdot(\dot{\mathbf{c}}_1\times\dot{\mathbf{d}}_1) - \mathbf{B}\cdot\dot{\mathbf{c}}_1 - \mathbf{A}\cdot\dot{\mathbf{d}}_1. \end{aligned}\tag{24}$$

Now we have to look at the vector part of (22). Using some vector algebra we get

$$\begin{aligned} \mathbf{b}_1 &= -\frac{1}{2}((\mathbf{m}+\mathbf{c}_1)^2-(\mathbf{n}+\mathbf{d}_1)^2)(\dot{\mathbf{c}}_1\times\dot{\mathbf{d}}_1) + (\mathbf{d}-\mathbf{c})\times((\mathbf{d}\cdot\dot{\mathbf{d}})\dot{\mathbf{c}} - (\mathbf{c}\cdot\dot{\mathbf{c}})\dot{\mathbf{d}})\\ &= -\frac{1}{2}\left(\mathbf{m}^2-\mathbf{n}^2+\mathbf{c}_1^2-\mathbf{d}_1^2\right)(\dot{\mathbf{c}}_1\times\dot{\mathbf{d}}_1) + (\mathbf{n}\cdot\dot{\mathbf{d}}_1)((\mathbf{n}-\mathbf{m})\times\dot{\mathbf{c}}_1-\mathbf{A})\\ &\quad -(\mathbf{m}\cdot\dot{\mathbf{c}}_1)((\mathbf{n}-\mathbf{m})\times\dot{\mathbf{d}}_1+\mathbf{B}) + (\mathbf{n}\times\mathbf{B})\times\dot{\mathbf{c}}_1 + (\mathbf{m}\times\mathbf{A})\times\dot{\mathbf{d}}_1. \end{aligned}\tag{25}$$

Finally, the bisector $B$ is parametrized by

$$\mathbf{b}(u,v) = \frac{1}{b_0(u,v)}\mathbf{b}_1(u,v).$$

Substituting trigonometric by rational quadratic functions we see that $B$ is a rational tensor product surface of degrees (2,2). One can prove that $B$ is of order $\leq 4$.

*Summary 5.1.* The bisector surface of line–circle and circle–circle in general position is a supercyclide.

### 5.2.1. *Approximation of Curve–Curve Bisectors*

What we have seen so far is that the computation of curve–curve bisectors is easy. If one wants to use standard CAD representations of relatively low degrees, approximations are necessary. One way would be to approximate the input curves by smoothly joined arc splines, possibly containing linear segments; see [14, 17, 21, 24, 28, 35]. As outlined above, the corresponding bisector surfaces are piecewise of maximal degrees (2,2). This was one reason that line–line, line–circle, and circle–circle bisectors are studied here in detail.

### 5.2.2. *Bisectors of Pipe Surfaces with Same Radii*

Let $F$ and $G$ be two oriented pipe surfaces with same radii $d$ and spine curves $C$ and $D$, respectively. Their surface normals shall point both outward or inward. From Section 3.3 we know that their bisector $B$ is computable as the bisector of their spine curves $C, D$, interpreted as (degenerate) offset surfaces at oriented distance $-d$.

To obtain low-degree approximations, one will, as above, approximate the center curves by arc splines. This means approximating the pipes by spline surfaces, composed of tori segments. Their bisector is then again a rational spline surface, composed of patches on supercyclides.

The self-bisector $B_F$ of a pipe surface $F$ is also computable as the self-bisector of the spine curve $C$ of $F$. This can be done by inserting $\mathbf{d}(v) = \mathbf{c}(v)$ into (22).

## 6. EXAMPLES OF POINT–SURFACE BISECTORS

With the help of results obtained in Section 4 we want to discuss some examples of low-degree point–surface bisectors. These can serve as basic algorithms for the bisector computation.

Since the bisector mapping generating the point–surface bisector is decomposable into (scaling $\circ$ polarity $\circ$ inversion), all those families of surfaces which are closed under inversion play a special role. This means that inversion maps a surface to an image surface belonging to the same family.

*Spheres and planes* form a family which is closed under inversion. The bisector of a point and a plane is a paraboloid of revolution; the bisector of a point and a sphere is also a quadric of revolution, namely an ellipsoid or two sheet hyperboloid, depending on whether the point lies inside or outside of the sphere. In any case we get nonruled quadrics, which are projectively equivalent. This reflects the fact that for computing the bisector we first apply an inversion and then a polarity, which is a general projective mapping into the dual space.

Dupin cyclides also form a family of surfaces, which is closed under inversion. These surfaces possess two families of circles $c_1$, $c_2$ as curvature lines. These circles lie in two pencils of planes $\pi_1, \pi_2$; let the axes of these pencils be denoted by $A_1$, $A_2$. One of these axes can be possibly at infinity. The circles $c_1$ intersect the axis $A_1$ in two (not necessary distinct and real) singular surface points. The same holds for the family $c_2$. The developable surface which is tangent to $F$ along a circle is always a cone or cylinder of revolution or a plane. The vertices of cones tangent to the circles of family $c_1$ lie on the axis $A_2$ and vice versa.

The algebraic order of $F$ is $\leq 4$ and the quartic surfaces $F$ possess the absolute conic $J$ as double curve. There is quite a large amount of literature on Dupin cyclides, including classical contributions and more modern ones; see [23, 25, 33] and the references therein.

If one interprets a straight line as a circle with center at infinity and radius $= \infty$, the above definition of Dupin cyclides also applies to cones and cylinders of revolution. In fact, certain

Dupin cyclides are the inverse images of cones and cylinders of revolution; see [23]. Thus, before studying point–Dupin cyclide bisectors we want to discuss point–cone bisectors.

### 6.1. Point–Circular Cone and Point–Circular Cylinder Bisectors

A circular cone (cone of revolution) or a cylinder of revolution $F$ is generated by rotating a line $L$ around an axis $A$, where either $L$ intersects $A$ or $L$ and $A$ are parallel. The curvature lines of $F$ are the generator lines and the circles in planes orthogonal to $A$. From decomposition (19) we know that the bisector surface of $O$ and $F$ is

$$B = \sigma \circ \pi \circ \kappa(F).$$

Since $\kappa$ preserves curvature lines, $\kappa(F)$ is a Dupin cyclide. These surfaces are self-dual in the sense that along each circle there is a cone of revolution (cylinder, plane) tangent to $\kappa(F)$. The polarity $\pi$ preserves this self-dual property, such that $B$ possesses two families of conics $b_1$, $b_2$ lying in pencils of planes $\varepsilon_1$, $\varepsilon_2$ with axes $A_1$, $A_2$.

Along each conic $b_1$ there is a quadratic cone $C_1$ tangent to $B$. Its vertex is on the axis $A_2$. Analogously, along each conic $b_2$ of the second family there is a quadratic cone $C_2$ with vertex on $A_1$, which is tangent to $B$. Thus, $B$ is a supercyclide; compare Sections 5.1 and 5.2. Since $\pi$ is a general projective mapping, $b_1$ and $b_2$ are no longer curvature lines, but they form a conjugate net of curves on $B$; see [7]. Their algebraic order and class is $\leq 4$.

We can find these properties also via parametrization (6). In particular, let $F$ be a cylinder of revolution, parametrized by

$$\mathbf{f}(u, v) = \mathbf{m} + r\mathbf{a}\cos u + r\mathbf{b}\sin u + v\mathbf{a} \times \mathbf{b} = \mathbf{m} + \mathbf{c}(u) + v\mathbf{n},$$

where $\mathbf{a}$ and $\mathbf{b}$ are orthogonal unit vectors and $\mathbf{n} = \mathbf{a} \times \mathbf{b}$. Let $g(u_0)$ be a fixed generator of $F$. The bisector surface of $O$ and $g(u_0)$ is a parabolic cylinder $B_1(u_0)$. The intersection of $B_1(u_0)$ with the normal plane to $F$ through $g(u_0)$ is a conic $\mathbf{b}(u_0, v) \subset B$. Since parametrization (26) is a quadratic polynomial in $v$, the conic $\mathbf{b}(u_0, v)$ is a parabola.

The second family of conics $\mathbf{b}(u, v_0) \subset B$ corresponds to bisectors $B_2$ with respect to $O$ and the circles on $F$. $B_2$ are quadratic cones with vertices on the axis $A$. The conics $\mathbf{b}(u, v_0)$ are contained in parallel planes which are perpendicular to $A$; see Fig. 10. Via (6), a parametrization of $B$ is found by

$$\mathbf{b}(u, v) = \mathbf{m} + v\mathbf{n} + \mathbf{c}(u)\left(\frac{\frac{1}{2}(r^2 - \mathbf{m}\cdot\mathbf{m} - v^2) - v(\mathbf{m}\cdot\mathbf{n})}{\mathbf{m}\cdot\mathbf{c}(u) + r^2}\right). \tag{26}$$

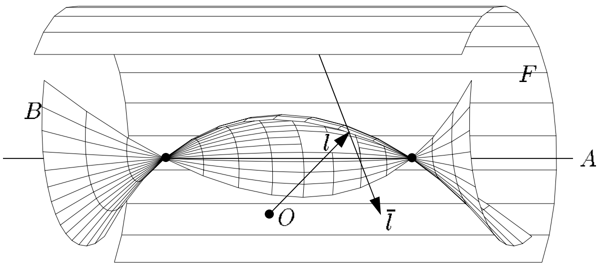


**FIG. 10.** Point–circular cylinder bisector.

Substituting trigonometric by rational functions leads to a (2,2) tensor product representation of $B$. Cones of revolution are calculated analogously.

### 6.2. Point–Dupin Cyclide Bisector

We use the arguments from Section 6.1. Those Dupin cyclides which are $\kappa$-images of cones or cylinders of revolution possess supercyclides as point–surface bisectors. It remains to discuss those Dupin cyclides which are $\kappa$-images of tori without real singular points.

A torus $T$ is generated by rotating a sphere $S$ around an axis $A$. The center of $S$ generates a circle $m$. $T$ possesses two families of circles $c_1$, $c_2$ in planes through $A$ and perpendicular to $A$. The developable surfaces tangent to $T$ along circles $c_1, c_2$ are cones or cylinders of revolution. We compute the bisector surfaces with respect to $c_1$, $c_2$ and obtain families of quadratic cones $B_1$, $B_2$. The characteristic curves on surfaces $B_1$ are intersections with planes through $A$ and thus are conics. The characteristic curves on surfaces $B_2$ are intersections with cones of revolution $N_2$ with axis $A$. The cones $N_2$ are orthogonal to $T$ along circles $c_2$. Since $B_2$ possesses $A$ as focal line (see Section 3.1), the two cones $B_2$ and $N_2$ possess common conjugate imaginary tangent planes through $A$. This implies that $B_2 \cap N_2$ is reducible, for instance, splits up into two (not necessarily distinct) conics.

A parametrization can be computed by using formula (6). Figure 11 shows an example. The real singular points of $B$ lie on the circle $m$. This figure also illustrates the mirror and anticaustic properties of $B$ and $T$ with respect to an illumination emanating from $O$.

*Summary 6.1.* The bisector surface $B$ of a point $O$ and a cone or cylinder of revolution or a Dupin cyclide $F$ is in general a supercyclide. Light rays $l$ radiating from a point source $O$ are reflected at $B$ in such a way that the reflected rays $\bar{l}$ are orthogonal to $F$.

### 6.3. Point–Quadric Bisector

The bisector surface $B$ of a regular quadric $F$ and a point $O$ shall be constructed. The decomposition of the bisector mapping (19) says that we first have to apply an inversion $\kappa$ to $F$ with respect to an arbitrary sphere $S$, centered at $O$. If $O$ is in general position to $F$ then $\kappa(F)$ is a general cyclide, a surface of order 4, possessing the absolute conic $J$ as double curve. Since a nonrevolutionary quadric possesses two 1-parameter families of real

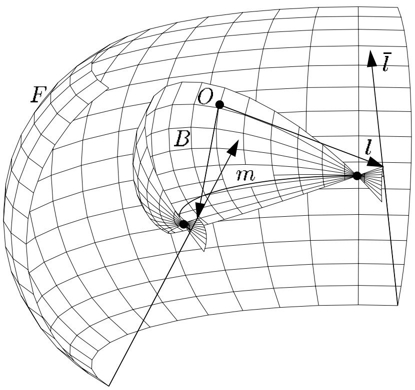


**FIG. 11.** Point–torus bisector.

circles, $\kappa(F)$ has this property, too. But note that these circles are never conjugate curves on $F$.

We apply the polarity $\pi$ with respect to $S$ and obtain a surface $\pi(\kappa(F))$ of class 4, possessing two 1-parameter families of quadratic cones, tangent to $\pi(\kappa(F))$. Further, scaling by $\frac{1}{2}$ does not change these properties.

A low-degree representation in dual (plane) coordinates can be obtained by parametrizing a triaxial quadric $F$ corresponding to its circular sections. These families of circles are mapped to circular sections of $\kappa(F)$. Finally, the bisector is representable as a rational (2,2) tensor product surface in plane coordinates. The point representation of the bisector is rational of degrees (4,4). Nonruled quadrics of revolution possess bisectors with a (2,4) tensor product representation in plane coordinates, since they possess only one family of circular sections.

Let the quadric $F$ be a quadratic cone, but not a cone of revolution. As above, $F$ contains two families of circles, which are never conjugate curves. The inversion $\kappa$ maps $F$ to a general cyclide $\kappa(F)$. Since $F$ is the envelope of a one-parameter family of planes, $\kappa(F)$ is a canal surface, which is enveloped by a one-parameter family of spheres. The generator lines as well as the circles of $F$ are mapped to families of circles on the cyclide $\kappa(F)$. Finally, the bisector surface possesses a rational (2,2) tensor product parametrization in plane coordinates. Its point representation is of degrees (4,2), where the quadratic parameter curves correspond to generator lines of $F$; see Section 6.5.

EXAMPLE. A general quadratic cone $F$ can be parametrized by

$$\mathbf{f}(u, v) = (a + vr\cos u, b + vs\sin u, c(1 - v)), \tag{27}$$

where $a, b, c$ are the coordinates of the vertex and $r, s$ are the major and minor axes of an elliptic intersection with the plane $z = 0$. It would be possible to set $r = s$ and to locate the vertex not at the rotational axis of the planar intersection with the plane $z = 0$.

Let $C$ be the intersection of $F$ with the plane at infinity $x_0 = 0$. To determine the circular sections of $F$ we have to compute the intersection points of $C$ and the absolute conic $J$, which are given by equations

$$C : \frac{1}{r^2}x_1^2 + \frac{1}{s^2}x_2^2 - \frac{1}{c^2}x_3^2 = 0, \qquad J : x_1^2 + x_2^2 + x_3^2 = 0.$$

$C \cap J$ consists of two pairs of conjugate complex points $p, \overline{p}$ and $q, \overline{q}$. The real lines connecting $p, \overline{p}$ and $q, \overline{q}$ are carriers of pencils of parallel planes, which intersect $F$ in two families of circles. One of these two families shall be parameter curves. A reparametrization can be realized by substituting

$$v = \frac{w - c(b\sqrt{r^2 - s^2} - s\sqrt{c^2 + r^2})}{cs(\sin u\sqrt{r^2 - s^2} + \sqrt{c^2 + r^2})}$$

in formula (27). Real circles of $F$ are represented by $\mathbf{f}(u, w_0)$ for real values $w_0$. We will substitute trigonometric by rational functions, $\cos u = (1 - t^2)/(1 + t^2)$ and $\sin u = 2t/(1 + t^2)$. The rational parametrization of $F$ can be written as

$$\mathbf{f}(t, w) = \frac{1}{f_0}(f_1, f_2, f_3)(t, w),$$

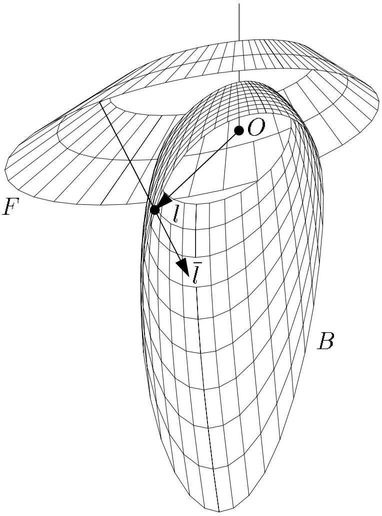


**FIG. 12.** Bisector of quadratic cone and point.

and the symmetry plane of $O$ and $\mathbf{f}$ is given by the equation

$$-\frac{1}{2}\left(f_1^2 + f_2^2 + f_3^2\right) + f_0(f_1 x + f_2 y + f_3 z) = 0.$$

It turns out that $f_0$ and $f_1^2 + f_2^2 + f_3^2$ possess common zeros, determined by $2t\sqrt{r^2 - s^2} + (1 + t^2)\sqrt{c^2 + r^2} = 0$. Further, we see that the denominator and numerator of $\mathbf{b}(t, w)$ possess the factor $cs\sqrt{c^2 + r^2} + w$. Applying these algebraic manipulations leads to a representation of degrees (4,2) of $B$ as a point set. The result is illustrated in Fig. 12.

A parametrization of skew-ruled quadrics can be based on two families of generator lines or it can be based on two families of circular sections. The first possibility leads to a rational (2,2) parametrization as a set of planes and to a rational (3,3) parametrization as a point set. Compare Section 6.4, where we discuss bisectors of points and skew-ruled surfaces. See also Fig. 13, which illustrates the bisector surface of a point and a hyperbolic paraboloid. The second way leads to a (2,2) representation in plane coordinates and further a rational (4,4) representation as a point set.

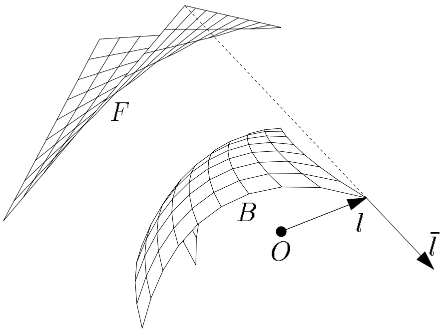


**FIG. 13.** Bisector of point and skew-ruled surface.

### 6.4. Point–Ruled Surface Bisector

Ruled surfaces are among the simplest surfaces, since one family of parameter curves consists of straight lines. A ruled surface $F$ possesses a parametrization

$$\mathbf{s}(u, v) = \mathbf{r}(u) + v\mathbf{l}(u),$$

where $\mathbf{r}(u)$ is a directrix curve and its rulings are given by vectors $\mathbf{l}(u)$. The surface normal vector is $(\dot{\mathbf{r}} + v\dot{\mathbf{l}}) \times \mathbf{l}$ and it shall really depend on $v$ and shall not be constant along a ruling. This excludes developable surfaces, which are studied in Section 6.5. The bisector surface of the origin $O$ and $F$ can then be parametrized by

$$\mathbf{b}(u, v) = \mathbf{r} + v\mathbf{l} - \frac{1}{2}\frac{(\mathbf{r} + v\mathbf{l})^2}{\det(\mathbf{r}, \dot{\mathbf{r}} + v\dot{\mathbf{l}}, \mathbf{l})}(\dot{\mathbf{r}} \times \mathbf{l} + v\dot{\mathbf{l}} \times \mathbf{l}). \tag{28}$$

Geometrically we can proceed as follows. The bisector $B_G(u_0)$ of the line $G(u_0) = \mathbf{r}(u_0) + v\mathbf{l}(u_0)$ and point $O$ is a parabolic cylinder. The surface normals of $F$ along the skew generator $G(u_0)$ form a hyperbolic paraboloid $H(u_0)$. The intersections $B_G(u_0) \cap H(u_0)$ are cubic parameter curves of $B$; see parametrization (28). This also follows from the fact that $B_G(u_0)$ and $H(u_0)$ possess a line at infinity in common. The cubics possess exactly one real point at infinity, since there is exactly one surface normal along $G(u_0)$, which is parallel to the generator lines of $B_G(u_0)$.

### 6.5. Point–Developable Surface Bisector

Let $C$ be a regular space curve with parametrization $\mathbf{c}(u)$. The tangent lines $\mathbf{c}(u) + \lambda\dot{\mathbf{c}}(u)$ of $C$ form a developable surface $F$ with parametrization

$$\mathbf{f}(u, v) = \mathbf{c}(u) + v\dot{\mathbf{c}}(u). \tag{29}$$

The curve $C$ is called the line of regression of the developable $F$. If $C$ is just a single point $\mathbf{z}$, $F$ is called a cone. One can use (29) but has to exchange $\mathbf{c}$ by $\mathbf{z}$ and $\dot{\mathbf{c}}(u)$ parametrizes the generator lines of the cone. General cylinders do not fit this parametrization since their vertex is at infinity. But one can use $\mathbf{f}(u, v) = \mathbf{c}(u) + v\mathbf{d}$ with a regular curve $C$ and a constant vector $\mathbf{d}$, parallel to the generators, but not parallel to $\dot{\mathbf{c}}$.

To determine the bisector $B$ of $O$ and developable surface $F$ with line of regression $C$, we first calculate the symmetry planes of $O$ and $\mathbf{f}(u, v)$,

$$\sigma(u, v) : \mathbf{x} \cdot (\mathbf{c}(u) + v\dot{\mathbf{c}}(u)) = \frac{1}{2}(\mathbf{c}(u)^2 + 2\mathbf{c}(u) \cdot \dot{\mathbf{c}}(u) + v^2\dot{\mathbf{c}}(u)^2).$$

For fixed $u_0$ planes $\sigma(u_0, v)$ form a quadratic family. These planes envelop the bisector $B_1(u_0)$ of a generating line $G(u_0) : \mathbf{c}(u_0) + v\dot{\mathbf{c}}(u_0)$ of $F$ and $O$. $B_1(u_0)$ is a parabolic cylinder. The intersection of $B_1(u_0)$ with the normal plane $N : (\dot{\mathbf{c}} \times \ddot{\mathbf{c}}) \cdot (\mathbf{x} - \mathbf{c}) = 0$ passing through the generator $G(u_0)$ is a parabola. A parametrization of $B$ is obtained by inserting (29) into (6),

$$\mathbf{b}(u, v) = \mathbf{c} + v\dot{\mathbf{c}} - \frac{1}{2}\frac{(\mathbf{c} + v\dot{\mathbf{c}})^2}{\det(\mathbf{c}, \dot{\mathbf{c}}, \ddot{\mathbf{c}})}\dot{\mathbf{c}} \times \ddot{\mathbf{c}}.$$

Parameter curves $\mathbf{b}(u_0, v)$ are the above-mentioned parabolas $\subset B$. In general, surfaces generated by one-parameter families of conics possess the property that tangent planes along a fixed conic envelop a rational developable surface of order 4; see [5]. In this special case, these developable surfaces along parameter parabolas $\mathbf{b}(u_0, v)$ are actually quadratic cones (parabolic cylinders).

## 7. SPHERE–SURFACE BISECTOR

We have already seen in Section 3.3 that sphere–surface bisectors are of a special kind. Let $S$ be an oriented sphere with radius $d$, centered at the origin $O$, and let $F$ be an arbitrary smooth oriented surface. The orientation of $S$ can also be determined by a signed radius $d$. If the oriented normals of $S$ point to the exterior of $S$, let $d$ be positive. The sphere–surface bisector $B$ of $S$ and $F$ is exactly the point–surface bisector of $O$ and $F_{-d}$, the one-sided offset surface of $F$ at oriented distance $-d$.

Let $\mathbf{f}(u, v)$ be a parametrization of $F$, and let $\mathbf{f}_u$ and $\mathbf{f}_v$ be the partial derivative vectors. The offset surface $F_d$ is parametrized by

$$\mathbf{f}_d(u, v) = \mathbf{f}(u, v) + d\mathbf{n}_0(u, v), \qquad \text{with } \mathbf{n}_0 = \frac{1}{\|\mathbf{f}_u \times \mathbf{f}_v\|}(\mathbf{f}_u \times \mathbf{f}_v),$$

where $\mathbf{n}_0$ is the oriented unit normal of $F$. It is known that an arbitrary rational surface $F$ will not possess rational offset surfaces.

We call $F$ a *rational PN surface* if there exists a parametrization $\mathbf{f}(u, v)$ such that the unit normal $\mathbf{n}_0$ is rational in $u, v$. A detailed study of these surfaces is given in [30]. We know several examples for such surfaces, for instance, spheres, cylinders, cones of revolution, and Dupin cyclides. Surprisingly all regular quadrics in 3-space are rational PN surfaces; see [22]. Further, it is known that all rational skew-ruled surfaces and canal surfaces with a rational spine curve and a rational radius function are rational PN surfaces. Additional examples can be found in [29].

*Summary 7.1.* The bisector surface $B$ of a rational PN surface $F$ and a sphere $S$ with radius $d$ centered at $O$ is a rational surface. A parametrization of $B$ is obtained by computing the bisector of $O$ and $F_{-d}$, the offset surface of $F$ at oriented distance $-d$.

### 7.1. Sphere–Dupin Cyclide Bisector

The families of Dupin cyclides, cones, and cylinders of revolution are closed under offsetting. This implies that the bisector surface of an oriented sphere and a Dupin cyclide, cone, or cylinder of revolution is a supercyclide, as studied in Sections 6.1 and 6.2. By the way, the axis of rotation of a cone or cylinder of revolution and the spine curve of a torus shall also be denoted as the “offset surface.”

### 7.2. Approximation of Sphere–Canal Surface Bisector

Let $F$ be an arbitrary canal surface and let $S$ be a sphere with radius $d$, centered at $O$. The offset surface $F_{-d}$ of $F$ is again a canal surface. Only its radius has decreased by $d$. We will approximate the canal surface $F$ or $F_{-d}$ by a sequence of Dupin cyclides; see [32]. Let $C_i$ be a sequence of circles on $F_{-d}$ and let $\Delta_i$ be tangent cones along them. For each pair $(C_i, \Delta_i)$ and $(C_{i+1}, \Delta_{i+1})$ of circle plus tangent cone one computes a pair of Dupin cyclides $D_i, D_{i+1}$ with the property that $D_i$ is tangent to $\Delta_i$ along $C_i$ and $D_{i+1}$ is tangent to $\Delta_{i+1}$ along $C_{i+1}$. Additionally, the Dupin cyclides $D_i$ and $D_{i+1}$ are tangent to each other along a circle.

The bisector $B$ of $F$ and $S$ is now approximated by the bisector surface $\overline{B}$ of the sphere $S$ and the sequence $D_i$ of Dupin cyclides. From Section 6.2 we know that $\overline{B}$ is composed of supercyclides and thus is a rational biquadratic spline surface.

## 8. CURVE–SURFACE BISECTOR

Let $C$ be a smooth curve and let $F$ be a smooth surface in 3-space. The curve–surface bisector $B$ contains all centers of spheres which are tangent to $C$ and $F$. Let $C$ and $F$ be parametrized by $\mathbf{c}(t)$ and $\mathbf{f}(u, v)$. Fixing a point $\mathbf{c}(t_0)$ on $C$, we will compute the bisector surface $B_F(t_0)$ of $\mathbf{c}(t_0)$ and $F$ with formula (6). Varying the parameter value $t_0$ we obtain a one-parameter family of bisector surfaces $B_F(t)$,

$$\mathbf{b}_F(u, v, t) = \mathbf{f}(u, v) - \frac{1}{2}\frac{(\mathbf{f} - \mathbf{c})^2}{\det(\mathbf{f} - \mathbf{c}, \mathbf{f}_u, \mathbf{f}_v)}\mathbf{f}_u \times \mathbf{f}_v, \tag{30}$$

which envelops the bisector surface $B$ of $F$ and $C$. Let $N(t_0)$ be the normal plane to $C$ at the curve point $\mathbf{c}(t_0)$. The parameter curves of the bisector $B$ are intersection curves

$$D(t) = B_F(t) \cap N(t).$$

The computation is nonlinear, and rational input data will not lead to rational bisector surfaces in general.

EXAMPLE. Given a Dupin cyclide or a cone of revolution $F$ and an arbitrary space curve $C$, the bisector $B_F(t)$ is a supercyclide for all points on $C$. Intersecting $B_F(t)$ with the normal plane $N(t): (\mathbf{x} - \mathbf{c}(t)) \cdot \dot{\mathbf{c}}(t) = 0$ leads to a polynomial equation, quadratic in $u$ and $v$. To obtain the intersection $D(t)$ one has to solve

$$(\mathbf{b}_F(u, v, t) - \mathbf{c}(t)) \cdot \dot{\mathbf{c}}(t) = 0 \tag{31}$$

for $v$ (or $u$), from which one obtains a parametrization $\mathbf{b}(u, t)$ (or $\mathbf{b}(v, t)$) of the bisector $B$. This involves square roots of the parameters $t, u$ (or $t, v$).

It has to be noted here that the bisector construction for an oriented pipe surface $G$ of radius $d$ and oriented surface $F$ can be translated into curve–surface bisector construction for the spine curve $C$ of $G$ and offset surface $F_{-d}$ to $F$.

One would expect a simplification of the bisector construction if the curve were a straight line. Unfortunately, elimination (31) is not significantly easier in this case. If $F$ is a developable surface, the computation is linear (up to some normalization). This shall be proved in Section 10.

In the following we will concentrate on pairs of curves $C$ and surfaces $F$ where the construction of the bisector $B$ is linear. Mainly this is the case when $F$ is a plane or a sphere, and $C$ is an arbitrary curve.

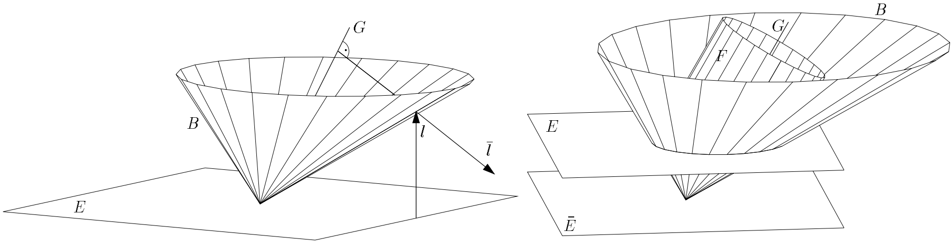


**FIG. 14.** Line–plane and cylinder–plane bisector.

### 8.1. Line–Plane Bisector

We will see in Section 10 that this is a special case of bisectors between developable surfaces. Nevertheless, since line–plane is a basic algorithm it shall be discussed here. Given a line $G$ and a plane $E$, if $G$ is parallel to $E$, their bisector $B$ is a parabolic cylinder with focal line $G$ and generators parallel to $G$. If $G$ is not parallel to $E$, let $V$ be the intersection point $G \cap E$. The bisector is a quadratic cone $C$ with vertex $V$; see Fig. 14 and [8, 10].

A parametrization is obtained as follows. Let $\mathbf{n} = (0, 0, 1)$ be the unit normal of $E$. We will assume that $V = G \cap E$ is the origin $O$ of a chosen coordinate system. Thus, $G$ is parametrized by $\mathbf{g} = \lambda(\cos\beta, 0, \sin\beta)$ with a real parameter $\lambda$. Let $\mathbf{k}$ and $\mathbf{m}$ be two unit normals of $\mathbf{g}$ with the property $\mathbf{k} \cdot \mathbf{m} = 0$. Without loss of generality we can assume that $\mathbf{m} = (0, 1, 0)$ and $\mathbf{k} = (-\sin\beta, 0, \cos\beta)$. The pencil of planes passing through $G$ can be parametrized by

$$\tau(t) : \mathbf{x} \cdot (\mathbf{k}\cos t + \mathbf{m}\sin t) = 0.$$

Since the plane $E$ is $\mathbf{x} \cdot \mathbf{n} = 0$, the symmetry planes of $E$ and $\tau(t)$ are

$$\sigma(t) : \mathbf{x} \cdot (\mathbf{n} - \mathbf{k}\cos t - \mathbf{m}\sin t) = 0.$$

These planes envelop the quadratic cone $B$, the bisector of $G$ and $E$. Inserting the above parametrization, the generator lines of $B$ are

$$b(t) = \lambda(-\cos\beta + \cos t, -\sin\beta\sin t, -\sin\beta).$$

Using a rational reparametrization for cos and sin we get a quadratic parametrization of $B$. If $G$ is orthogonal to $E$, $B$ is a cone of revolution otherwise a general quadratic cone. Its *focal lines* are $G$ and the normal to $E$ through $V$.

Consider light rays $l$ orthogonal to a plane $E$. We look for a mirror surface $B$, such that the reflected rays $\bar{l}$ intersect a given line $G$ orthogonally. If $G$ is not perpendicular to the light rays $l$, $B$ is one of the cones with vertex on $G$. All these cones are translational versions of one another, translated along $G$. If $G$ is perpendicular to the light rays $l$, $B$ is a parabolic cylinder.

The inverse problem is also of interest. Given an illumination $l$ orthogonal to a cylinder of revolution $F$ with axis $G$, we search for a mirror surface $B$, such that the reflected light rays $\bar{l}$ are parallel, or say normal to a plane $E$. The solution is again a parabolic cylinder in the case of $G$ parallel to $E$. Otherwise the mirror can be chosen to be the mentioned cone $B$; see Fig. 14.

### 8.2. Curve–Plane Bisector

Given a plane $F$ and a curve $C$, parametrized by $\mathbf{c}(u)$, the construction of the bisector $B$ can be done by calculating the envelope of the family $B_F(u)$, where $B_F(u)$ is the bisector of $F$ and a variable point $\mathbf{c}(u)$ on $C$. We know that $B_F(u)$ are paraboloids of revolution, with $\mathbf{c}(u)$ as the focal points. The characteristic curves $d(u)$, which are the curves of contact of $B$ and $B_F(u)$, are conics, namely the planar intersections

$$d(u) = B_F(u) \cap N(u).$$

$N(u)$ is considered to be the normal plane of $C$ at $\mathbf{c}(u)$. Practically, let $F$ be the plane $z=0$. The bisector of a point $\mathbf{c}(u)$ and $F$ is given by the equation

$$B_F(u)\colon 2c_3(u)z = c_3(u)^2 + (x - c_1(u))^2 + (y - c_2(u))^2.$$

In this case the derivative $\dot{B}_F(u)$ and the normal plane $N(u)$ are identical,

$$N(u)\colon \dot{c}_3(u)z = c_3(u)\dot{c}_3(u) - \dot{c}_1(u)(x - c_1(u)) - \dot{c}_2(u)(y - c_2(u)). \tag{32}$$

The orthogonal projections $d_1(u)$ of the conics $d(u)$ onto $F$ are circles with the equation

$$d_1(u)\colon \left(x - \frac{c_1\dot{c}_3 - c_3\dot{c}_1}{\dot{c}_3}\right)^2 + \left(y - \frac{c_2\dot{c}_3 - c_3\dot{c}_2}{\dot{c}_3}\right)^2 = \frac{c_3^2}{\dot{c}_3^2}\left(\dot{c}_1^2 + \dot{c}_2^2 + \dot{c}_3^2\right). \tag{33}$$

Let $\mathbf{m}(u) = (m_1, m_2, 0)(u)$ be the center of $d_1(u)$. Then it is easy to prove that $\mathbf{m}(u)$ is the intersection point of the tangent line $T(u) : \mathbf{c}(u) + \lambda\dot{\mathbf{c}}(u)$ with $F$.

This leads to another generation of $B$. We consider the bisector surface $B_T(u)$ of $F$ and a tangent line $T(u)$ of $\mathbf{c}(u)$, which is a quadratic cone with vertex $\mathbf{m}(u)$. $B_T(u)$ is tangent to $B_F(u)$ exactly in the characteristic conics $d(u)$ and we have $d(u) = B_T(u) \cap N(u)$; see Fig. 15.

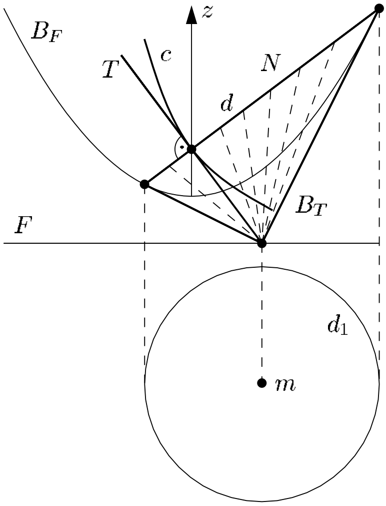


**FIG. 15.** Front and top view of plane–curve bisector construction.

In particular, let $C$ be a polynomial or rational curve. Then $B_F(u)$ and $N(u)$ are rational families of paraboloids and planes. In particular, $d(u)$ and $d_1(u)$ are rational families. The center curve $\mathbf{m}(u)$ of $d_1(u)$ is rational and the radius function of these circles is a square root of a rational function; see Eq. (33).

It is known that $d(u)$ generates a rational surface, and an algorithm to compute a rational parametrization is described in [29]. We just want to note that the algorithm mainly requires the calculation of all zeros of $(\dot{c}_1^2 + \dot{c}_2^2 + \dot{c}_3^2)(u)$ in the case of a polynomial curve or the zeros of its numerator otherwise. Rational parametrizations of $B$ are then obtained by solving a linear system.

*Summary 8.1.* The curve–plane bisector $B$ for rational curves is a rational surface, enveloped by a one-parameter family of paraboloids of revolution $B_F(u)$ or quadratic cones $B_T(u)$. One family of parameter curves comprises conics which are curves of contact of $B_F(u)$ and $B_T(u)$.

### 8.3. Circle–Plane Bisector

We want to discuss this example here, since the construction of the circle–plane bisector is linear (Fig. 16). From an algorithmic viewpoint, any smooth spatial curve $K$ can be approximated by circular arc segments; see [17, 21, 35]. Thus, the circle–plane bisector can serve as a basic algorithm for the approximation of general curve–plane bisectors. Given the plane $F : z = 0$ and a circle $C$, which can be parametrized by

$$\mathbf{c}(u) = \mathbf{n} + r\,\mathbf{a}\cos u + r\,\mathbf{b}\sin u,$$

We can assume that

$$\mathbf{n} = (0, n\cos\phi, n\sin\phi), \qquad \mathbf{a} = ((0, \cos\phi, \sin\phi)), \qquad \mathbf{b} = (1, 0, 0).$$

The intersection line of $F$ and the plane carrying $C$ is the $x$-axis. The family of circles $d_1(u)$ is given by

$$d_1(u) : \left(x - \frac{r + n\cos u}{\sin u}\right)^2 + y^2 = \left(\frac{n + r\cos u}{\sin u}\right)^2.$$

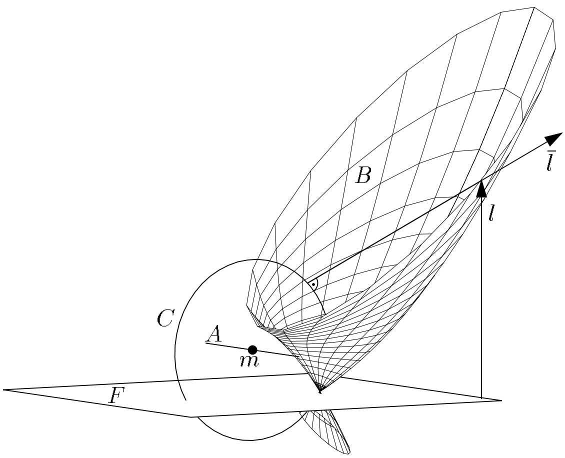


**FIG. 16.** Plane–circle bisector.

These circles possess two common points $(0, \pm\sqrt{n^2 - r^2}, 0)$, which are conjugate complex or real, depending on whether $C$ intersects $F$ in real points or not. By the way, these points are the top views of singular surface points, which lie on the rotational axis $A$ of $C$. Further, the real or conjugate complex intersection points of $C$ and $F$ are also singular points on $B$. A parametrization of the bisector surface $B$ is

$$\begin{aligned} b_1(u, v) &= \frac{r + n \cos u}{\sin u} + \frac{n + r \cos u}{\sin u} \cos v, \\ b_2(u, v) &= \frac{n + r \cos u}{\sin u} \sin v, \\ b_3(u, v) &= -\frac{1}{\sin \phi (\sin u)^2} (n + r \cos u)(-1 + \cos u \cos v - \sin u \cos \phi \sin v), \end{aligned} \tag{34}$$

where $(b_1(u, v), b_2(u, v))$ is a parametrization of circles $d_1(u)$ and $b_3(u, v)$ is calculated via formula (32). It turns out that $B$ is in general a surface of order 4 with equation

$$B : (x^2 + y^2)^2 + p_3(x, y) + z p_2(x, y) + c z^2 = 0,$$

where $p_i$ are polynomials of degree $i$ and $c$ is a constant. These surfaces are called isotropic cyclides and their singular curve of order 2 degenerates to two conjugate complex lines in the plane at infinity. They are non-Euclidean counterparts of envelopes of quadratic families of spheres; see [18]. Using rational instead of trigonometric functions in (34) results in a rational tensor product representation of degrees (4,2) in $u, v$.

## 8.4. Curve–Sphere Bisectors

Given a sphere $S$ of radius $d$, centered at $O$, and a regular (spatial) curve $C$, parametrized by $\mathbf{c}(t)$, we can use the offset surface property of the bisector $B$, which says that $B$ is also the bisector of $O$ and the pipe surface $F$ with center curve $C$ and radius $d$. This leads to the same construction as the direct way, which shall be outlined here.

The bisector $B_S(t_0)$ of a fixed point $\mathbf{c}(t_0) \notin S$ of $C$ is a quadric of revolution with focal points $O$ and $\mathbf{c}(t_0)$. The bisector $B$ of $S$ and $C$ contains parameter conics

$$d(t_0) = B_S(t_0) \cap N(t_0),$$

where $N(t_0)$ is the normal plane to $C$ at $\mathbf{c}(t_0)$; see Fig. 17. The tangent planes of $B$ along $d(t_0)$ form a quadratic cone $D(t_0)$ with vertex $V(t_0)$ on the tangent line of $C$ in $\mathbf{c}(t_0)$. The bisector $B$ is the envelope of a one-parameter family of quadrics $B_S(t_0)$. Let $\Delta(t_0)$ be the cone with vertex $O$ through $d(t_0)$. Intersecting $\Delta(t_0)$ with $S$ gives a circle $d_S(t_0)$. The pole to $d_S(t_0)$ with respect to $S$ is the vertex $V(t_0)$ of $D(t_0)$.

Assume that $C$ is a rational curve. Then $B_S(t)$ is a rational family of quadrics of revolution and $N(t)$ is a rational family of planes. This implies that $d(t)$ is a rational family of conics which generate a rational surface $B$. The computational effort to get exact rational parametrizations of $B$ is the same as that to obtain rational parametrizations of the pipe surface $F$ with spine curve $\mathbf{c}$ and then calculate the bisector of $F$ and $O$.

*Summary 8.2.* The curve–sphere bisector $B$ of a rational curve $C$ and sphere $S$ is a rational surface, enveloped by a one-parameter family of quadrics of revolution $B_S(t)$ or quadratic cones $D(t)$. One family of parameter curves consists of conics, the curves of contact of $B_S(t)$ and $D(t)$.

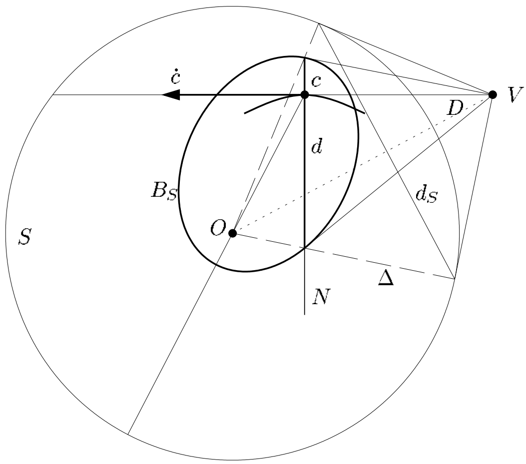


**FIG. 17.** Principle of curve–sphere bisector.

## 9. SURFACE–SURFACE BISECTOR

The construction of the bisector of two surfaces $F$ and $G$ is in general complicated. The simplest example in this context is the bisector of two oriented planes $F\colon f_0 + \mathbf{f} \cdot \mathbf{x} = 0$ and $G\colon g_0 + \mathbf{g} \cdot \mathbf{x} = 0$. We assume that the normal vectors $\mathbf{f}$ and $\mathbf{g}$ are normalized. The bisector of $F$ and $G$ is the unique plane of symmetry

$$\sigma : (f_0 - g_0) + (\mathbf{f} - \mathbf{g}) \cdot \mathbf{x} = 0.$$

If one surface is a sphere, see the results of Section 7. If one surface is a pipe surface, see Section 8. If both surfaces $F$ and $G$ are developable, see Section 10. Otherwise, the only basic example in view of computational treatment is the plane–surface bisector, which shall be discussed here.

### 9.1. Plane–Surface Bisector

Given an oriented nondevelopable surface $F$ and an oriented plane $A$, let $\mathbf{f}(u, v)$ be a parametrization of $F$ and let $\mathbf{n}_0(u, v)$ be its unit normals. The tangent planes of $F$ are

$$\tau(u, v) : \mathbf{n}_0(u, v) \cdot (\mathbf{x} - \mathbf{f}(u, v)) = 0.$$

We assume that $A$ contains the origin $O$ and is given by an equation $\mathbf{a} \cdot \mathbf{x} = 0$, with $\mathbf{a} \cdot \mathbf{a} = 1$. The symmetry planes $\sigma(u, v)$ of $\tau$ and $A$ are

$$\sigma(u, v) : \mathbf{n}_0 \cdot \mathbf{f} + (\mathbf{a} - \mathbf{n}_0) \cdot \mathbf{x} = 0$$

and envelop the bisector surface $B$ of $F$ and $A$, which means that $\sigma(u, v)$ is a dual parametrization of $B$. A point representation of $B$ is found by intersecting $\sigma(u, v)$ with the surface normal $\mathbf{f}(u, v) + \lambda \mathbf{n}(u, v)$. This leads to

$$\mathbf{b}(u, v) = \mathbf{f}(u, v) - \frac{(\mathbf{a}_0 \cdot \mathbf{f})}{\mathbf{n}_0 \cdot (\mathbf{a} - \mathbf{n}_0)} \mathbf{n}_0.$$

In the case of a rational surface $F$ the bisector is not necessarily rational. If $F$ is a rational

*PN* surface, which means $F$ possesses a parametrization such that the unit normals $\mathbf{n}_0$ are rational, then the plane–surface bisector $B$ possesses a rational parametrization $\mathbf{b}(u, v)$.

The optical interpretation of the bisector construction tells us that parallel light rays $l$, which are orthogonal to $A$, are reflected at $B$ in such a way that the reflected light rays $\overline{l}$ are orthogonal to $F$. So, $F$ can be called *anticaustic* with respect to the given parallel illumination and reflection at the mirror surface $B$.

*Summary 9.1.* The bisector of a plane and a rational *PN* surface is a rational surface.

## 10. BISECTORS OF DEVELOPABLE SURFACES

Nonplanar developable surfaces are envelopes of one-parameter families of tangent planes. One obtains three different types, namely cylinders, cones, and surfaces formed by the tangent lines of space curves. These developable surfaces carry a one-parameter family of generating lines. The calculation of the bisector works as follows. Let $\Phi$ and $\Psi$ be two developable surfaces, both described by their one-parameter families of tangent planes

$$\Phi : F(u) \colon f_0(u) + \mathbf{f}(u) \cdot \mathbf{x} = 0,$$
$$\Psi : G(v) \colon g_0(v) + \mathbf{g}(v) \cdot \mathbf{x} = 0.$$

The generating lines of $\Phi$ are obtained by intersecting the planes $F(u)$ and $\dot{F}(u)$, the derivative plane of $F(u)$. Analogously for the generators of $\Psi$, where $\dot{G}(v)$ denotes the derivative of $G(v)$ with respect to $v$. Let $p(u) = F(u) \cap \dot{F}(u)$ be a generator of $\Phi$. All spheres tangent to $\Phi$ in points of $p(u)$ have to have centers lying in the normal plane $N_f(u)$ to $F(u)$ through $p(u)$. We can assume that the normal vectors $\mathbf{f}$ and $\mathbf{g}$ of $F$ and $G$ are normalized, $\|\mathbf{f}\| = 1$ and $\|\mathbf{g}\| = 1$. We obtain

$$N_f(u) \colon \dot{f}_0(u) + \dot{\mathbf{f}}(u) \cdot \mathbf{x} = 0,$$
$$N_g(v) \colon \dot{g}_0(v) + \dot{\mathbf{g}}(v) \cdot \mathbf{x} = 0,$$

and note that the dots denote derivatives with respect to $u$ and $v$. There will be no confusion since the functions $f_i$ depend only on $u$, whereas functions $g_i$ depend only on $v$.

A sphere, tangent to $\Phi$ in some point of $p(u)$ and tangent to $\Psi$ in some point of $q(v) = G \cap \dot{G}$, has to have its center in the symmetry plane $S(u, v)$ of $F(u)$ and $G(v)$. If $\Phi$ and $\Psi$ are oriented surfaces, their tangent planes can be oriented by the oriented unit normals $\mathbf{f}(u)$ and $\mathbf{g}(u)$. So, the computation of $S$ is unique and we get

$$S(u, v) \colon (f_0 - g_0) + (\mathbf{f} - \mathbf{g}) \cdot \mathbf{x} = 0. \tag{35}$$

We see that the computation of the bisector surface $B$ of two oriented developables $\Phi$ and $\Psi$ is a linear problem, up to the normalization of $\mathbf{f}$ and $\mathbf{g}$. A parametrization $\mathbf{b}(u, v)$ of $B$ is found by solving the linear system

$$N_f(u) \cap N_g(v) \cap S(u, v). \tag{36}$$

In a closed form we can write the parametrization $\mathbf{b}(u, v)$ of $B$ as

$$\mathbf{b}(u, v) = \frac{1}{\mathbf{A} \cdot \dot{\mathbf{g}} + \mathbf{B} \cdot \dot{\mathbf{f}}}(-\dot{g}_0\mathbf{A} - \dot{f}_0\mathbf{B} + (\dot{g}_0\mathbf{g} - g_0\dot{\mathbf{g}}) \times \dot{\mathbf{f}} + (\dot{f}_0\mathbf{f} - f_0\dot{\mathbf{f}}) \times \dot{\mathbf{g}}), \tag{37}$$

where $\mathbf{A} = \mathbf{f} \times \dot{\mathbf{f}}$ and $\mathbf{B} = \mathbf{g} \times \dot{\mathbf{g}}$. If the emphasis is on rational parametrizations we can state the following.

*Summary 10.1.* Let $\Phi$ and $\Psi$ be two oriented developable surfaces which possess rational unit normals $\mathbf{f}(u)$ and $\mathbf{g}(v)$. Then the above construction proves that the bisector surface $B$ is rational.

The most important examples of developable surfaces with rational unit normals are cones and cylinders of revolution. More generally, the normal vector has to be of the form

$$\mathbf{f}(u) = \frac{1}{a(u)^2 + b(u)^2 + c(u)^2}(2a(u)c(u), 2b(u)c(u), a(u)^2 + b(u)^2 - c(u)^2), \quad (38)$$

with relatively prime polynomials $a(u), b(u), c(u)$. Using an arbitrary rational function $f_0(u)$ one obtains a one-parameter family of planes

$$F(u)\colon f_0(u) + \mathbf{x} \cdot \mathbf{f}(u) = 0.$$

Its envelope is a developable surface with rational unit normals $\mathbf{f}(u)$.

### 10.1. Bisector of Cylinders of Revolution

Given two cylinders of revolution $\Phi$ and $\Psi$ with tangent planes $F(u)$ and $G(v)$, respectively, let $C$, $D$ be circular cross sections of $\Phi$, $\Psi$. These circles can be parametrized by

$$C: \mathbf{m} + r\mathbf{f}(u), \qquad \text{with } \mathbf{f} = \mathbf{a}\cos u + \mathbf{b}\sin u,$$
$$D: \mathbf{n} + s\mathbf{g}(v), \qquad \text{with } \mathbf{g} = \mathbf{c}\cos v + \mathbf{d}\sin v,$$

where the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{d}$ are normalized, and $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{c} \cdot \mathbf{d} = 0$. Thus, $\mathbf{f}$ and $\mathbf{g}$ are normalized. Tangent planes and derivative planes of $\Phi$ and $\Psi$ are

$$F(u): (\mathbf{x} - \mathbf{m} - r\mathbf{f}(u)) \cdot \mathbf{f}(u) = 0, \qquad \dot{F}(u) : (\mathbf{x} - \mathbf{m}) \cdot \dot{\mathbf{f}}(u) = 0,$$
$$G(v): (\mathbf{x} - \mathbf{n} - s\mathbf{g}(v)) \cdot \mathbf{g}(v) = 0, \qquad \dot{G}(v) : (\mathbf{x} - \mathbf{n}) \cdot \dot{\mathbf{g}}(v) = 0.$$

The symmetry plane of the oriented planes $F(u)$ and $G(v)$ is

$$S(u, v): (-\mathbf{m} \cdot \mathbf{f} + \mathbf{n} \cdot \mathbf{g} - r + s) + (\mathbf{f} - \mathbf{g}) \cdot \mathbf{x} = 0.$$

Then, an elementary calculation leads to a parametrization of $B$,

$$\mathbf{b}(u, v) = \frac{1}{\mathbf{A} \cdot \dot{\mathbf{g}} + \mathbf{B} \cdot \dot{\mathbf{f}}}(\mathbf{n} \cdot \dot{\mathbf{g}}\mathbf{A} + \mathbf{m} \cdot \dot{\mathbf{f}}\mathbf{B} + (\mathbf{A} \times \mathbf{m}) \times \dot{\mathbf{g}} + (\mathbf{B} \times \mathbf{n}) \times \dot{\mathbf{f}} + (r - s)\dot{\mathbf{f}} \times \dot{\mathbf{g}}), \quad (39)$$

with $\mathbf{A} = \mathbf{f} \times \dot{\mathbf{f}}$ and $\mathbf{B} = \mathbf{g} \times \dot{\mathbf{g}}$ as constant vectors representing the axes of the cylinders. Substituting trigonometric by rational functions one obtains a tensor product representation of degrees (2,2) for $B$. This says that the parameter curves are conics. These two families of conics are contained in pencils of planes passing through the axes of the cylinders $\mathbf{m} + \lambda\mathbf{A}$ and $\mathbf{n} + \mu\mathbf{B}$, respectively. The intersection points of $B$ with these two axes are singular points of $B$. The bisector surface is a supercyclide, and its double curve is a pair of lines in the plane at infinity.

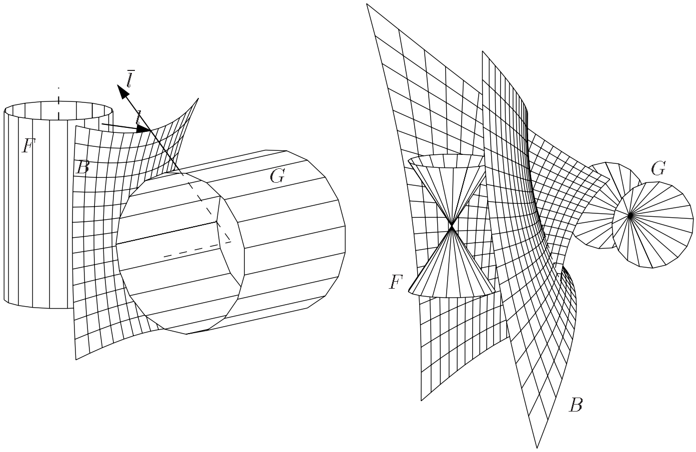


**FIG. 18.** Bisector surfaces of two cones and cylinders of revolution.

### 10.2. Bisector of Two Cones of Revolution

Given two cones of revolution $\Phi$ and $\Psi$ with vertices $\mathbf{m}$ and $\mathbf{n}$ and tangent planes $F(u)$ and $G(v)$, respectively, the normal vectors of these tangent planes can be represented by

$$\mathbf{f}(u) = \frac{1}{\sqrt{1+r^2}}(\mathbf{a}\times\mathbf{b} + r\mathbf{a}\cos u + r\mathbf{b}\sin u),$$
$$\mathbf{g}(v) = \frac{1}{\sqrt{1+s^2}}(\mathbf{c}\times\mathbf{d} + s\mathbf{c}\cos v + s\mathbf{d}\sin v),$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{d}$ are normalized and $\mathbf{a}\cdot\mathbf{b} = 0$ and $\mathbf{c}\cdot\mathbf{d} = 0$ (Fig. 18). Then, $F$ and $G$ and their derivatives are

$$F(u): (\mathbf{x}-\mathbf{m})\cdot\mathbf{f}(u) = 0, \qquad \dot{F}(u): (\mathbf{x}-\mathbf{m})\cdot\dot{\mathbf{f}}(u) = 0,$$
$$G(v): (\mathbf{x}-\mathbf{n})\cdot\mathbf{g}(v) = 0, \qquad \dot{G}(v): (\mathbf{x}-\mathbf{n})\cdot\dot{\mathbf{g}}(v) = 0.$$

Let $S = F - G$ be the symmetry plane of $F$ and $G$. The bisector of $\Phi$ and $\Psi$ is parametrized by

$$\mathbf{b}(u,v) = \frac{1}{\mathbf{A}\cdot\dot{\mathbf{g}} + \mathbf{B}\cdot\dot{\mathbf{f}}}(\mathbf{n}\cdot\dot{\mathbf{g}}\mathbf{A} + \mathbf{m}\cdot\dot{\mathbf{f}}\mathbf{B} + (\mathbf{A}\times\mathbf{m})\times\dot{\mathbf{g}} + (\mathbf{B}\times\mathbf{n})\times\dot{\mathbf{f}}), \tag{40}$$

with $\mathbf{A} = \mathbf{f}\times\dot{\mathbf{f}}$ and $\mathbf{B} = \mathbf{g}\times\dot{\mathbf{g}}$. Here, $\mathbf{A}$ and $\mathbf{B}$ are not constant. But if we insert the above representations for $\mathbf{f}$ and $\mathbf{g}$ and substitute the trigonometric by rational functions, one can verify that (40) is a (2,2) tensor product representation of $B$. The bisector $B$ is a supercyclide which has singular points on the axes of the cones and a pair of lines as singular curve.

*Summary 10.2.* The bisectors of cones and cylinders of revolution are supercyclides.

*Remark.* The cyclographic model applies quite well to the bisector construction for cones of revolution; see [19]. The cyclographic image of the cones $F$, $G$ are straight lines $f$, $g$ in 4-space. The cyclographic images of all spheres tangent to $F$, $G$ form quadratic hypercones $\Gamma_f$ and $\Gamma_g$ with $f$ and $g$, respectively, as singular sets. The intersection $D = \Gamma_f \cap \Gamma_g$

is a two-dimensional surface of order 4 in 4-space, and its orthogonal projection onto $E^3 : r = 0$ is the bisector $B$ of $F$ and $G$.

The hypercones $\Gamma_f$, $\Gamma_g$ contain generating planes $\phi$ and $\psi$ passing through $f$ and $g$, respectively. The orthogonal projections of $\Gamma_f \cap \psi$ and $\Gamma_g \cap \phi$ are the two families of conics on $B$. Moreover, the intersection points $f \cap \Gamma_g$ and $g \cap \Gamma_f$ are projected onto the singular points on the axes of $F$ and $G$.

## 11. CONCLUSION

This article shall enlighten the role of classical geometry in the computation of bisector surfaces. It is also a collection of basic algorithms and linear constructions. The general surface–surface bisector construction is not linear, but for several surface families, such as spheres, pipe surfaces, and developable surfaces, we have found elementary methods.

This article is mainly a geometrical contribution and in view of algorithms, a lot of work has to be done, since the spatial problems seem to be much more complicated than the planar ones.

We obtained similar results for point–surface, sphere–surface, and plane–surface bisectors, as well as for bisectors of two curves or developable surfaces. We have seen that certain families of curves and surfaces play a special role. They share the property of invariance under Möbius transformations. Since Laguerre geometric properties have also occurred, a Lie sphere geometric investigation would be a unifying method for bisector constructions. This will be studied in a further contribution.

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