J. Symbolic Computation (1997) **23**, 255–266

Computing Rational Parametrizations of Canal Surfaces

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(Received 11 December 1995)

A canal surface is the envelope of a one-parameter set of spheres with radii $r(t)$ and centers $m(t)$. It is shown that any canal surface to a rational spine curve $m(t)$ and a rational radius function $r(t)$ possesses rational parametrizations. We derive algorithms for the computation of these parametrizations and put particular emphasis on low degree representations.

1. Introduction

Current CAD systems can represent curves and surfaces only in rational B-spline (NURBS) form (Farin, 1994; Hoschek and Lasser, 1993). On the other hand, certain curves and surfaces that arise in practical applications such as offsets of rational curves or surfaces are in general not rational and therefore need to be approximated. This motivated Farouki and Sakkalis (1990) to introduce the so-called *Pythagorean-hodograph (PH) curves*, which are planar polynomial curves that possess rational offsets. Recent research on PH curves and their generalizations to the full class of rational curves with rational offsets has shown that they are well–suited for practical use (see e.g. Ait Haddou and Biard, 1995; Albrecht and Farouki, 1995; Farouki, 1992; Lü, 1995a; Pottmann, 1995a, b and the references therein).

The offset at distance r to a curve $m(t)$ in 3-space can be defined as the envelope of the set of spheres with radius r which are centered at $m(t)$. Such a surface is called a pipe surface or tubular surface with spine curve $m(t)$. Surprisingly, it turned out that *pipe surfaces with rational spine curve $m(t)$ always admit a rational parameterization* (Lü and Pottmann, 1996). In the present paper, we will generalize this result as follows. *Canal surfaces, defined as envelope of a one-parameter set of spheres with a rational radius function $r(t)$ and centers at a rational curve $m(t)$ can be rationally parametrized.* A constructive proof for this result is given, along with other techniques to compute rational parametrizations of pipe and canal surfaces with low degree spine curves and radius functions. In practical applications, canal surfaces mainly appear as blend surfaces and transition surfaces between pipes. Note that the present class of surfaces contains as special case the Dupin cyclides, which have been proposed by several authors for various applications in Computer Aided Geometric Design (see e.g. Pratt, 1995; Srinivas and Dutta, 1994).

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2. Geometric Properties of Canal Surfaces

Let $\mathbb{R}^3$ be Euclidean 3-space with Cartesian coordinates x_1, x_2, x_3. A point p is represented with respect to a coordinate system by a vector (p_1, p_2, p_3), and in general we do not distinguish between the point and its coordinate vector. Let $\mathbb{P}^3$ be projective 3-space. A point p is represented by its homogeneous coordinates $p = (\bar{p}_0 : \bar{p}_1 : \bar{p}_2 : \bar{p}_3)$. If $\bar{p}_0 \neq 0$ the relation between Cartesian and homogeneous coordinates is $p_i = \bar{p}_i/\bar{p}_0$ for $i = 1, 2, 3$. In general x denotes a Cartesian coordinate vector or point in 3-space. By the map $x \to (1, x)$ we embed $\mathbb{R}^3$ into $\mathbb{P}^3$. Then the complement of $\mathbb{R}^3$ in $\mathbb{P}^3$ is the plane at infinity or *ideal plane* and the homogeneous coordinates of its points satisfy $x_0 = 0$. The Euclidean scalar product of two vectors a, b shall be denoted by $a \cdot b$, the vector product by $a \times b$.

A canal surface Φ is defined as envelope of a one-parameter set of spheres $\Sigma(t)$, centered at a *spine curve* $m(t)$. The radius of the spheres is given by the function $r(t)$, $t \in \mathbb{R}$. The defining equations for Φ are

$$\Sigma(t) : (x - m(t))^2 - r(t)^2 = 0, \tag{2.1}$$

$$\dot{\Sigma}(t) : (x - m(t)) \cdot \dot{m}(t) + r(t)\dot{r}(t) = 0. \tag{2.2}$$

A canal surface Φ contains a one parameter set of so called *characteristic* circles $k(t) = \Sigma(t) \cap \dot{\Sigma}(t)$. Obviously the plane $\dot{\Sigma}$ is perpendicular to the derivative vector $\dot{m}$. Elimination of the parameter t from above equations leads to an equation of Φ. Figure 1 illustrates that a canal surface Φ can also be interpreted as envelope of a one parameter set of cones of revolution $\Delta(t)$. These cones are tangent to Φ at points of the characteristic circles $k(t)$.

Let q be the unit normals of Φ. For each fixed t_0 the vector field $q(t_0, u)$ shall represent the unit normals at points of $k(t_0)$, such that a parametric representation of Φ is given by

$$\Phi : x(t, u) = m(t) + r(t)q(t, u). \tag{2.3}$$

If $r \equiv$ const., Φ is the envelope of a moving sphere and is called a pipe surface. A pipe surface can also be interpreted as the envelope of a one parameter set of congruent cylinders of revolution. The plane $\dot{\Sigma}$ intersects Σ in a great circle, centered at m. This implies that a pipe surface is always real.

The reality of a canal surface Φ depends in general on $\dot{r}$ and the length of $\dot{m}$. Substituting $y = x - m$ in (2.1) and (2.2), it follows that $(y \cdot \dot{m})^2 = y^2\dot{r}^2$. One obtains

$$1 \geq \cos^2 \alpha = \frac{(y \cdot \dot{m})^2}{y^2 \dot{m}^2} = \frac{\dot{r}^2}{\dot{m}^2}. \tag{2.4}$$

We conclude that the envelope is real, exactly if $\dot{m}^2 - \dot{r}^2 \geq 0$.

Remark: If equality holds for a parameter value t_0, the plane $\dot{\Sigma}(t_0)$ is tangent to the sphere $\Sigma(t_0)$, such that $k(t_0)$ degenerates to a single point. If equality holds in a nonempty interval, the envelope degenerates to a curve plus the one parameter set of tangent planes $\dot{\Sigma}(t)$.

In the following equality shall hold only for isolated parameter values. For the construction of parametrizations it is necessary that (2.4) is true for all real numbers. In cases where the reality condition is satisfied only for $t \in [a, b] \neq \mathbb{R}$ and $a \neq b$, one may use a reparametrization. For instance, let $t = (a + bs^2)/(1 + s^2)$, such that the reality condition holds for all $s \in \mathbb{R}$.

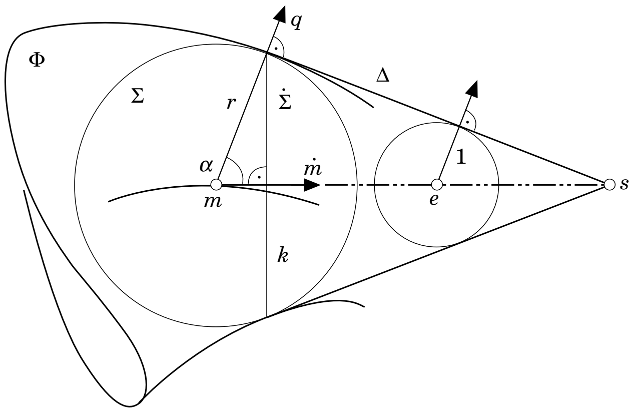

Figure 1. Geometric properties.

Since rationality is quite important for practical use, we only deal with rational spine curves $m(t)$ and rational radius functions $r(t)$. We introduce two methods deriving rational parametrizations of the form (2.3). In both cases the main problem is to find a rational curve contained in Φ. We did a first attempt in Section 3. But the necessary conditions lead to calculations, which are rather difficult. The second one (Section 5) is more geometric and essentially uses the Gauss map. Further we give a constructive proof of the existence of rational parametrizations of real canal surfaces determined by a rational spine curve $m(t)$ and a rational radius $r(t)$.

3. System of Quadratic Equations

Let $m(t)$ be a rational spine curve and $r(t)$ a rational radius function. A rational parametrization of the form (2.3) is given, if it is possible to construct a rational vector function $q(t,u)$ such that $x = m + rq$ satisfies (2.1) and (2.2). The resulting conditions are $q^2 \equiv 1$ and

$$q \cdot \dot{m} + \dot{r} \equiv 0, \tag{3.1}$$

corresponding to equations (2.1) and (2.2). Each characteristic circle k can be parametrized in a rational way by a parameter u, described in Section 3.2. The main problem is to determine a unit vector field $\tilde{q}(t)$ which satisfies (3.1), such that $f = m + r\tilde{q}$ is a rational curve in Φ.

Since $\tilde{q}(t)$ is a rational curve contained in the unit sphere S^2, it follows from Dietz *et al.* (1993) that a homogeneous coordinate representation of $\tilde{q}$ is

$$\begin{aligned} \tilde{q}_0 &= p_0^2 + p_1^2 + p_2^2 + p_3^2, & \tilde{q}_1 &= 2(p_0p_1 - p_2p_3), \\ \tilde{q}_2 &= 2(p_0p_2 + p_1p_3), & \tilde{q}_3 &= p_0^2 - p_1^2 - p_2^2 + p_3^2, \end{aligned}$$

with polynomials $p_0(t), \ldots, p_3(t)$. Assume that the Cartesian coordinate functions m_i and the radius function r are of degree $\le k$ and have a common denominator d. Let $p_i(t) = p_{i0} + p_{i1}t + \cdots + p_{in}t^n$ be polynomials of degree n. Then equation (3.1) is a polynomial of degree $2n + 2k - 2$. The identity condition in (3.1) leads to a system of $2n+2k-1$ equations, quadratic in the $4(n+1)$ homogeneous unknowns p_{ij} for $i = 0, 1, 2, 3$ and $j = 0, \ldots, n$.

If $n \ge k - 2$, it would be possible that (3.1) has real solutions. Further the rational

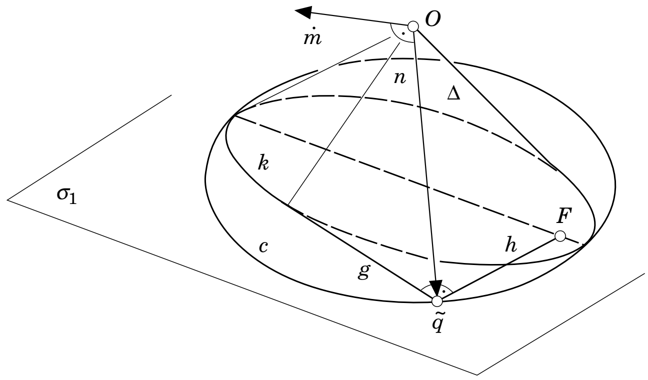

Figure 2. A geometric method for pipe surfaces to a polynomial cubic spine curve.

vector field $\tilde{q}$ would be of degree $2k-4$. For example, let $k=3$, and thus $n \geq 1$. Identity (3.1) results in a system of 5 polynomial equations in 8 homogeneous unknowns. For this case under the condition of special radius functions we are able to prove that real solutions exist for $n=k-2$. But in genereral it is not obvious how to choose an appropriate n, for which real solutions exist.

However, a real solution of (3.1) defines a rational curve $f = m + r\tilde{q}$ contained in Φ. The remaining calculations to obtain the complete parametrization (2.3) are given in Section 3.2. In special cases, solving system (3.1) can be avoided. Several examples are discussed in Section 3.1 and Section 4.

3.1. A GEOMETRIC METHOD FOR PIPE SURFACES

Let Φ be a pipe surface determined by a cubic polynomial space curve $m(t)$. We describe a geometric method to find a unit normal vector field $\tilde{q}(t)$, which satisfies $\tilde{q}\cdot\dot{m} \equiv 0$. Since $m(t)$ is a cubic polynomial space curve, $\dot{m}(t)$ is quadratic and $x(\lambda, t) = \lambda\dot{m}(t)$ is a parametrization of a quadratic cone with vertex at the origin O. This implies that the planes $n(t) : x \cdot \dot{m} = 0$ are tangent planes of a quadratic cone, say Δ. A parametric representation of Δ is given by $x(\lambda, t) = \lambda(\dot{m} \times \ddot{m})(t)$. The quadratic form corresponding to Δ can be found by eliminating λ and t from the parametric representation. Each quadratic cone possesses at least three planes of symmetry, corresponding to the eigenvalues of the quadratic form. One plane of symmetry, say σ, intersects Δ in two conjugate complex lines. It follows that each plane $\sigma_1 \neq \sigma$, parallel to σ, intersects Δ in an ellipse k. The tangent lines $g(t)$ of k are obtained by $g = n \cap \sigma_1$ (see Figure 2).

Our aim is to construct a quadratic unit vector field $\tilde{q}(t)$, which is contained in $n(t)$. Let c be the circle in σ_1, tangent to the ellipse k at its main vertices. Let $h(t)$ be a pencil of lines in σ_1, passing through one focal point F of k, such that $h(t)$ is perpendicular to $g(t)$ for each $t \in \mathbb{R}$. It can be verified that the point $g(t) \cap h(t)$ is contained in c for each t. This construction determines a rational quadratic parametrization of c.

So it follows that the vector $\tilde{q}(t)$, which describes the point $g(t) \cap h(t) \subset \sigma_1$ satisfies $\tilde{q} \cdot \dot{m} \equiv 0$. Further $\tilde{q}(t)$ is of constant length and can be scaled to $\tilde{q}^2 \equiv 1$. The vector field $\tilde{q}(t)$ forms the basis of a rational parametrization of the pipe surface Φ. The remaining steps, that are the computation of the characteristic circles, can be done as described in Section 3.2.

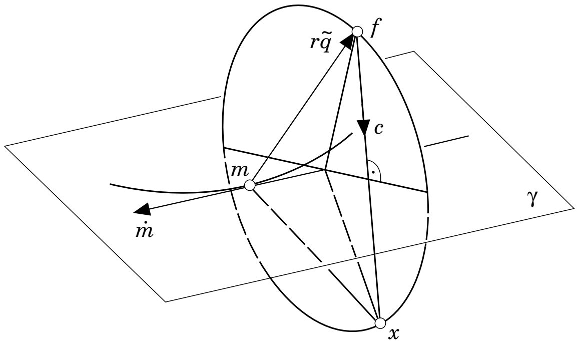

Figure 3. Computing the characteristic circles.

3.2. COMPUTING THE CHARACTERISTIC CIRCLES

Let $m(t)$ and $r(t)$ be rational and assume that $\tilde{q}(t)$ is already computed and defines a rational curve $f = m + r\tilde{q}$, contained in Φ. For a fixed t_0 let $\gamma(t_0, u)$ be the pencil of planes passing through the tangent line $m(t_0) + \lambda\dot{m}(t_0)$, as illustrated in Figure 3. Further let $c(t_0, u)$ be the normal vectors of $\gamma(t_0, u)$ given by

$$c(t_0, u) = v_1(t_0) + uv_2(t_0),$$

where v_1, v_2 denote two distinct normal vectors of planes contained in this pencil. We may choose v_1, v_2 in the following way

$$v_1 = (\dot{m}_2, -\dot{m}_1, 0), \qquad v_2 = (\dot{m}_3, 0, -\dot{m}_1).$$

Note that singularities may occur, namely for parameter values where v_1, v_2 are linearly dependent. This can be avoided by $c(t, u) = \dot{m}(t) \times n(t) + un(t)$, where $n(t)$ denotes a normal vector field of the spine curve $m(t)$.

One generates the characteristic circle corresponding to t_0 by reflecting $f(t_0)$ at all planes $\gamma(t_0, u)$. This construction leads to the following parametrization of Φ, namely

$$x(t, u) = f(t) - 2\frac{r(t)c(t, u) \cdot \tilde{q}(t)}{c(t, u)^2}c(t, u). \tag{3.2}$$

We mention that the parameter u varies in $\mathbb{R} \cup \infty$.

4. Canal Surfaces Tangent to Special Surfaces

In this section several examples shall be presented, where it is not necessary to solve (3.1), since a rational curve $f \subset \Phi$, different from a characteristic circle is already given. It follows from (3.2) that Φ is rational. To obtain parametric representations of Φ one only has to apply the construction described in Section 3.2 or similar ones.

First we study the class of canal surfaces tangent to a constant plane. This class includes well-known surfaces like Dupin cyclides. A rational curve f contained in Φ is the orthogonal projection of the spine curve m onto the constant tangent plane.

Let $m(t)$ be an arbitrary rational curve. Let $x_3 = 0$ be the constant tangent plane. This implies $r(t) = m_3(t)$ and with (3.2) the canal surface can be parametrized by

$$x(t, u) = (m_1, m_2, 0) + 2\frac{c_3 m_3}{c_1^2 + c_2^2 + c_3^2}(c_1, c_2, c_3). \tag{4.1}$$

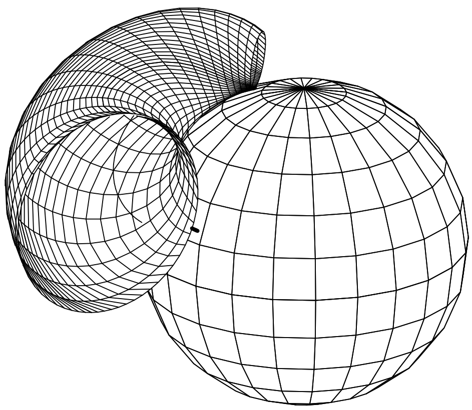

Figure 4. Canal surface tangent to a sphere.

A special case of the first example are pipe surfaces with planar spine curves. A parametric representation can be derived from (4.1) inserting constant radius functions.

A second example is canal surfaces tangent to a sphere along a rational curve. Let S^2 be the unit sphere, centered at the origin of 3-space. Let $f(t)$ be a rational curve $\subset S^2$, that means $f^2 \equiv 1$. Further let $r(t)$ be a rational radius function. Then the spine curve $m = rf$ and the radius r determine a rational canal surface Φ and a possible parametrization is

$$x(t,u) = f + \frac{2r^2(r-1)u}{1+u^2(r^2\dot{f}^2+\dot{r}^2)}(f \times \dot{f} + u(r\dot{f}^2 f - \dot{r}\dot{f})). \tag{4.2}$$

A special case of the second example are pipe surfaces with spherical spine curve. A parametrization is given with (4.2) for constant radius functions.

More general examples can be obtained in the following way. Let Ψ be a rational surface which possesses rational offsets and let $f(t)$ be a rational curve in Ψ. Then there is a rational unit vector field $n(t)$, perpendicular to Ψ in points of f. Let $r(t)$ be a rational function and $m = f + rn$ be a rational curve. Then it follows that the canal surface Φ determined by the spine curve m and the radius function r is rational and is tangent to Ψ along f.

5. Construction of Rational Parametrizations

In this section a general construction of rational parameterizations for real canal surfaces will be given. Further we prove that such parametrizations always exist.

Let Φ be a real canal surface, defined by a rational spine curve $m(t)$ and a rational radius function $r(t)$. Let Φ be interpreted as an envelope of a one parameter set of real cones of revolution $\Delta(t)$ (Figure 1). Let $s(t)$ be the vertices of $\Delta(t)$ and $e(t)$ be a curve, such that for each t_0 the sphere with center $e(t_0)$ and radius 1 is tangent to $\Delta(t_0)$. It follows that

$$s = m - \frac{r}{\dot{r}}\dot{m}, \qquad e = m + \frac{1-r}{\dot{r}}\dot{m}.$$

Let S^2 be the unit sphere centered at the origin of 3-space and let $\gamma : \Phi \to S^2$ be the Gauss map. The cones $\Delta(t)$ are mapped onto circles $c(t)$. The circles $c(t)$ themselves define cones of revolution $\tilde{\Delta}(t)$, which are tangent to S^2 along $c(t)$. The vertices of the cones $\tilde{\Delta}$ are given by $z = s - e$ which leads to

$$z(t) = \frac{-1}{\dot{r}(t)}\dot{m}(t).$$

Note that z is at infinity, if $\dot{r} = 0$.

Our aim is to construct a rational unit normal $q \subset S^2$ of Φ, such that (2.3) is a rational parametrization of Φ. For each fixed t_0 the vector field $q(t_0, u)$ describes a circle $\subset S^2$. To derive parametrizations it is comfortable to use a stereographic projection.

Let π be the Euclidean plane in 3-space defined by $x_3 = 0$ and $W = (0, 0, 1)$. A stereographic projection $\delta : S^2 \to \pi$ with center W is a rational conformal map. In particular δ maps circles c to circles or lines $\delta(c)$. In general $\delta(c)$ is a circle with center $n = \delta(z)$ and radius ρ given by

$$n = \frac{-1}{\dot{r} + \dot{m}_3}(\dot{m}_1, \dot{m}_2, 0), \tag{5.1}$$

$$\rho^2 = \frac{1}{(\dot{r} + \dot{m}_3)^2}(\dot{m}^2 - \dot{r}^2). \tag{5.2}$$

It is clear that $\delta(c)$ is a line, exactly if $\dot{r} + \dot{m}_3 = 0$. For further calculations one uses a Lemma, which can be proved by factorizing the given polynomial over the complex field.

Lemma 5.1. *Let f be a real definite polynomial, which means that $f(t) \geq 0$ for all $t \in \mathbb{R}$. Then there exist polynomials f_1, f_2, such that $f = f_1^2 + f_2^2$.*

Let us summarize what we have done till now. The stereographic projection of the Gauss image of Φ contains a one parameter set of circles $\delta(c)$, centered at a rational planar curve $n(t)$. But in general the radius function ρ is not rational. A rational curve $\tilde{\varphi}$ corresponding to the set $\delta(\tilde{c})$ shall be constructed, such that for each fixed t_0 the point $\tilde{\varphi}(t_0)$ is contained in the circle $\delta(c(t_0))$. Therefore $\tilde{\varphi}$ has to be of the form

$$\tilde{\varphi}(t) = n(t) + g(t), \tag{5.3}$$

where $g(t)$ a rational planar vector, whose coordinates satisfy $g_1^2 + g_2^2 = \rho^2$.

Since the denominator in (5.2) is a square, it is sufficient to apply Lemma 5.1 to the numerator of $\rho^2 \geq 0$. A solution g_1, g_2 of the decomposition of ρ^2 leads to the representation (5.3). To derive the complete parametrization of $\delta(\gamma(\Phi))$ one may proceed as follows. Let $d(u) = (u, 1)$ be normals of a pencil of lines. Similar to Section 3.2, a rational parametrization of a fixed circle $\delta(c)$ can be constructed by reflecting $\tilde{\varphi}$ at all diameters of $\delta(c)$, which leads to

$$\varphi(t, u) = \tilde{\varphi}(t) - 2\frac{g(t) \cdot d(u)}{d(u)^2}\, d(u). \tag{5.4}$$

The inverse projection $\delta^{-1} : \pi \to S^2$ maps φ to the unit normals

$$q(t, u) = \frac{1}{(1 + \varphi_1^2 + \varphi_2^2)}(2\varphi_1, 2\varphi_2, \varphi_1^2 + \varphi_2^2 - 1)(t, u), \tag{5.5}$$

such that $x(t, u) = m(t) + r(t)q(t, u)$ is a rational parametrization of Φ. Let us collect the derived results.

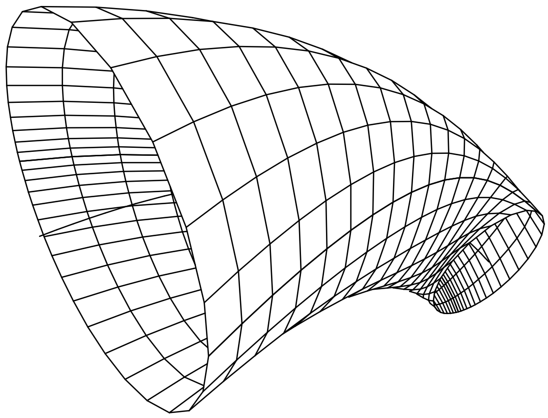

Figure 5. Canal surface defined by a cubic polynomial spine curve and a cubic polynomial radius.

THEOREM 5.1. *A real canal surface determined by a rational spine curve and a rational radius function possesses real rational parametrizations.*

REMARK: Theorem 5.1 is not a characterization of rational canal surfaces, but a sufficient condition. Additionally it admits a generalization on envelopes of rational one parameter sets of cones of revolution. Several examples and a proof are described by Peternell and Pottmann (1996).

In a further section practical calculations are studied in detail.

6. Degree Reductions and Implementation

The method derived in Section 5 depends on the choice of the center W of the stereographic projection $\delta : S^2 \to \pi$. The plane π is determined by the center W, because it has to be parallel to the tangent plane to S^2 at W. Otherwise, δ is not a conformal map. Equivalent to the choice of W is the choice of an adapted coordinate system. On the other hand low degree representations are important for practical use.

Let O be the origin and ξ, η and ζ unit vectors describing an orthonormal basis in $\mathbb{R}^3$. We will show that an adapted coordinate system $(O; \xi, \eta, \zeta)$ exists such that the degree of the parametrization of Φ reduces.

The map $\delta^{-1} : \pi \to S^2$ is quadratic, such that a rational planar curve $\varphi \subset \pi$ of order n is mapped to a rational curve $q \subset S^2$ of order $2n$ in general. The degree of q reduces, if the numerators and the denominator of the coordinate functions in (5.5) have a common divisor.

The description of the construction is easier if we assume that the coordinate functions $m_i(t)$ and the radius function $r(t)$ are polynomials. Otherwise we assume that the rational functions m_i and r have a common denominator d. Then formulae (6.4) and (6.5) can be used by replacing $\dot{m}_i$ by $d\dot{m}_i - \dot{d}m_i$ and so on.

6.1. APPROPRIATE CHOICE OF THE COORDINATE SYSTEM

Let ω be the ideal plane in $\mathbb{P}^3$, the projective extension of $\mathbb{R}^3$. We use $\mathbb{R}^3$ as Euclidean 3-space but also as vector space. A coordinate system or frame in $\mathbb{R}^3$ defines a coordinate

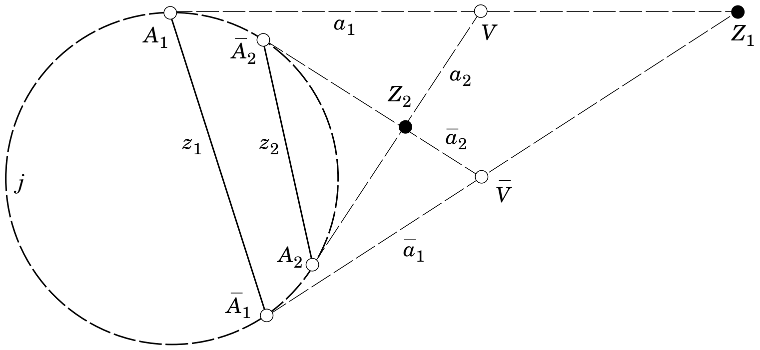

Figure 6. Appropriate choice of the coordinate system.

system in ω. A one-dimensional subspace λv, with $\lambda \neq 0$ and $v \neq (0,0,0)$ determines a point V in ω. We may interprete $\lambda(v_1, v_2, v_3)$ as homogeneous Cartesian coordinates of V in ω. Let j be the conic in ω defined by $x_1^2+x_2^2+x_3^2=0$. The bilinear form corresponding to j is the Euclidean scalar product $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$. The construction of the frame, illustrated in Figure 6, depends essentially on a configuration in ω. Complex lines are represented by dashed lines, real lines by solid lines. Complex points are represented by circles, real points by filled circles.

Let $\tau, \bar{\tau}$ be conjugate complex zeros or a real double zero of the definite polynomial $f = \dot{m}^2 - \dot{r}^2 \geq 0$, which is the numerator of (5.2). Let $v = \dot{m}(\tau)$ and $\bar{v} = \dot{m}(\bar{\tau})$. The vectors v and $\bar{v}$ describe conjugate complex points $V, \bar{V}$ in ω. Let a_i, $\bar{a}_i$ for $i = 1, 2$ be conjugate complex tangent lines of j, passing through V and $\bar{V}$. These lines are tangent to j in points A_i, $\bar{A}_i$. A coordinate representation is

$$A_i = (\alpha_i, \beta_i, \gamma_i) = (-v_1v_3 \mp \mathrm{i}\, v_2\sqrt{\lambda}, -v_2v_3 \pm \mathrm{i}\, v_1\sqrt{\lambda}, v_1^2 + v_2^2), \tag{6.1}$$

$$\bar{A}_i = (\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) = (-\bar{v}_1\bar{v}_3 \pm \mathrm{i}\, \bar{v}_2\sqrt{\bar{\lambda}}, -\bar{v}_2\bar{v}_3 \mp \mathrm{i}\, \bar{v}_1\sqrt{\bar{\lambda}}, \bar{v}_1^2 + \bar{v}_2^2), \tag{6.2}$$

where $\lambda = v \cdot v$ and $\bar{\lambda} = \bar{v} \cdot \bar{v}$. Using the scalar product it follows that the lines a_i and $\bar{a}_i$ are represented by the same coordinates as A_i and $\bar{A}_i$. That means that a_i for instance is given by the linear equation $\alpha_i x_1 + \beta_i x_2 + \gamma_i x_3 = 0$. Further let $Z_i = a_i \cap \bar{a}_i$ and let z_i be the lines connecting $A_i, \bar{A}_i$. It is clear that Z_i and z_i are real. We choose for instance the pair Z_1, z_1 and denote it for simplicity by Z, z. Analogously $A, \bar{A}$ denote the points $A_1, \bar{A}_1$. The point Z and the line z can be represented by the unit vector

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) = \frac{A \times \bar{A}}{\|A \times \bar{A}\|}. \tag{6.3}$$

The new coordinate system is chosen such that ζ describes the new x_3-axis. Let ξ and η be unit vectors of the new axes x_1, x_2. They can be chosen arbitrarily, but have to satisfy the conditions of an orthonormal frame. This implies $\eta = \zeta \times \xi$, where ξ is for instance

$$\xi_1 = \frac{\zeta_2}{\sqrt{\zeta_1^2 + \zeta_2^2}}, \qquad \xi_2 = -\frac{\zeta_1}{\sqrt{\zeta_1^2 + \zeta_2^2}}, \qquad \xi_3 = 0.$$

Since this construction depends only on a configuration in ω, the transformation is only determined up to sign changes of the basis vectors. The signs will be determined later.

6.2. DEGREE REDUCTIONS

First we rewrite the formulae of Section 5 in terms of m_i and r, assumed to be polynomials. Let f_1, f_2 be polynomials satisfying $f_1^2 + f_2^2 = \dot{m}^2 - \dot{r}^2$. Then the planar rational curve $\tilde{\varphi}$ is given by

$$\tilde{\varphi}(t) = \frac{1}{\dot{r} + \dot{m}_3}(f_1 - \dot{m}_1, f_2 - \dot{m}_2). \tag{6.4}$$

Using the following substitutions

$$e = \dot{m}_1^2 + \dot{m}_2^2, \qquad k = \dot{r} + \dot{m}_3, \qquad g = f_1\dot{m}_1 - f_2\dot{m}_2, \qquad h = f_1\dot{m}_2 + f_2\dot{m}_1,$$

and applying (5.4) and (5.5) the homogeneous coordinates of the unit normal q are

$$\begin{aligned}
\bar{q}_0(t,u) &= u^2(e + \dot{m}_3 k + g) + 2uh + e + \dot{m}_3 k - g,\\
\bar{q}_1(t,u) &= (-u^2(\dot{m}_1 + f_1) - 2uf_2 + f_1 - \dot{m}_1)k,\\
\bar{q}_2(t,u) &= (-u^2(\dot{m}_2 - f_2) - 2uf_1 - f_2 - \dot{m}_2)k,\\
\bar{q}_3(t,u) &= u^2(e - \dot{r}k + g) + 2uh + e - \dot{r}k - g.
\end{aligned} \tag{6.5}$$

We use the same notation as in Section 6.1 and assume that a frame is chosen as described there, up to the signs of the basis vectors. Let $d = (t-\tau)(t-\bar{\tau})$ and let $\pi : x_3 = 0$. We will show that d is a common divisor of the coordinate functions $\bar{q}_0, \ldots, \bar{q}_3$. The orthogonal projection $p : \mathbb{P}^3 \to \pi$ with center Z induces a planar projection $p_\omega : \omega \to z$ in the ideal plane. The projection p maps $v = \dot{m}(\tau)$ to $p(v)$ and $\bar{v} = \dot{m}(\bar{\tau})$ to $p(\bar{v})$; p_ω maps $V, \bar{V}$ to $A, \bar{A} \subset j$. Since $p(v)$ and $p(\bar{v})$ describe the points A and $\bar{A}$, which are contained in j, it follows that $p(v)^2 = p(\bar{v})^2 = 0$. We see that the polynomial $e = p(\dot{m})^2$ has zeros at τ and $\bar{\tau}$, such that d divides e. Since τ and $\bar{\tau}$ are also zeros of $\dot{m}^2 - \dot{r}^2$ it follows that d divides $\dot{m}_3^2 - \dot{r}^2$. We choose the orientation of ζ in (6.3) such that $\dot{m}_3(\tau) = -\dot{r}(\tau)$. This guarantees that d divides k. Maybe after a substitution of f_2 by $-f_2$ we achieve that the real polynomial d divides the complex polynomial $(\dot{m}_1 + \mathrm{i}\,\dot{m}_2)(f_1 + \mathrm{i}\,f_2)$. Therefore d also divides its real and imaginary parts, g and h, respectively.

We summarize that all coordinate functions $\bar{q}_i$ have a common divisor, such that the degree of the Cartesian coordinates $q_i = \bar{q}_i/\bar{q}_0$ reduces.

COROLLARY 6.1. *Let Φ be a real canal surface determined by a polynomial spine curve $m(t)$ of degree k and a polynomial radius function $r(t)$ of degree k. Then there exists a unit normal vector function $q(t,u)$ of Φ, which is of degree $2k-4$ in t. The resulting parametrization $(m+rq)(t,u)$ of Φ is in general of degree $3k-4$ in t and 2 in u.*

Let Φ be defined by rational but not polynomial functions $m_i(t)$ and $r(t)$ both of degree k, which possess a common denominator. The numerators of $\dot{m}$ and $\dot{r}$ are of degree $2k-2$ and it follows that the normal field $q(t,u)$ is of degree $4k-6$ in t. The resulting parametrization $(m+rq)(t,u)$ of Φ is in general of degree $5k-6$ in t and 2 in u.

Several problems occur when implementing the algorithm given above. One of them is the decomposition described in Lemma 5.1. In general one has to use numerical methods to calculate the zeros of f such that the solution f_1, f_2 is not exact. Further it is clear that $(t-\tau)(t-\bar{\tau})$ is not an exact divisor of the occuring polynomials. So it is necessary to combine algebraic and numerical methods (Stetter, 1996).

A further problem is the distribution of the rational parameter lines $f(t,u_0) = (m + rq)(t,u_0)$ for a fixed u_0 on a canal surface Φ (see Figures 5 and 7). Let Φ be a pipe

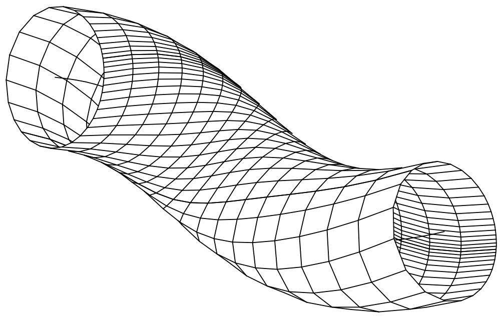

Figure 7. Pipe surface defined by a cubic polynomial spine curve.

surface. In general the distance between two fixed rational parameter lines $f(t, u_1)$ and $f(t, u_2)$, measured along the characteristic circles, is not constant. This fact can produce rational parametrizations, which are nearly singular. To avoid this one can restrict $m(t)$ to be contained in the class of curves, whose tangent vector $\dot{m}(t)$ has rational length (see Farouki and Sakkalis, 1994).

Finally, we would like to mention that Lü (1995b) has presented a different proof of the theorem along with another algorithm to construct rational parametrizations of canal surfaces. His method leads in general to higher degrees. Another contribution on this topic is a paper by Malosse (1996). He mainly studies pipe surfaces and the generalization to canal surfaces is not really straightforward. Furthermore we believe that our algorithm is easier to understand and to implement.

Acknowledgement

This work has been supported by the Austrian Science Foundation through project P09790–MAT.

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