On Generalized LN-Surfaces in 4-Space

Martin Peternell
University of Technology
Vienna, Austria
martin@geometrie.tuwien.ac.at

Boris Odehnal
University of Technology
Vienna, Austria
boris@geometrie.tuwien.ac.at

ABSTRACT

The present paper investigates a class of two-dimensional rational surfaces Φ in $\mathbb{R}^4$ whose tangent planes satisfy the following property: For any three-space E in $\mathbb{R}^4$ there exists a unique tangent plane T of Φ which is parallel to E. The most interesting families of surfaces are constructed explicitly and geometric properties of these surfaces are derived. Quadratically parameterized surfaces in $\mathbb{R}^4$ occur as special cases. This construction generalizes the concept of LN-surfaces in $\mathbb{R}^3$ to two-dimensional surfaces in $\mathbb{R}^4$.

Categories and Subject Descriptors

I.1 [**Symbolic and Algebraic Manipulation**]: Miscellaneous

General Terms

Theory

Keywords

LN-surface, quadratically parameterized surface, linear congruence of lines, chordal variety, rational parameterization.

1. INTRODUCTION

In $\mathbb{R}^3$ there exists a remarkable class of rational surfaces which are characterized by possessing a field of normal vectors which is linear in the surface parameters. These so-called LN-surfaces [2, 3] possess remarkable properties. Their family of tangent planes are represented by graphs of rational functions [6]. LN-surfaces possess rational offset surfaces [3] and the convolution surface of an LN-surface with an arbitrary rational non-developable surface is always rational [11]. Surprisingly, quadratically parameterized surfaces belong to this class [6].

This article will generalize the concept of LN-surfaces to $\mathbb{R}^4$. From the dual representation the generalization to hypersurfaces in $\mathbb{R}^4$ is evident, see Section 2.1. Interesting questions occur when determining two-dimensional rational surfaces Φ in $\mathbb{R}^4$ with similar properties. We characterize the class of rational surfaces Φ in $\mathbb{R}^4$ which satisfy the property that for all given 3-spaces E the surface parameters of Φ can be expressed in terms of *rational functions* of the coefficients of E. This implies that for all 3-spaces E there exist a unique tangent plane T parallel to E with a unique point of contact $\mathbf{p} \in \Phi$. The proposed construction presented in Section 3 circumvents the integration and leads to explicit parameterizations and geometric characterizations of these surfaces Φ. Quadratically parameterized surfaces in $\mathbb{R}^4$ occur as special cases.

The motivation for this research is based on the following relation: Considering $\mathbb{R}^4$ as model space of oriented spheres in $\mathbb{R}^3$, it has been shown in [7] that quadratically parameterized surfaces in $\mathbb{R}^4$ correspond to two-parameter families of spheres in $\mathbb{R}^3$ whose envelope surfaces and their offsets are rational surfaces. A criterion for all parameterized surfaces in $\mathbb{R}^4$ whose corresponding two-parameter families of spheres have envelopes which admit rational parameterizations has been given in [4].

As already indicated in [7], the proposed construction to obtain rational parameterizations of the envelopes of the two-parameter families of spheres can not only be performed with quadratically parameterized surfaces but with a much larger class of surfaces in $\mathbb{R}^4$. These surfaces are investigated here without taking into account these relations to sphere geometry but only considering the generalization of the concept of LN-surfaces to $\mathbb{R}^4$.

The paper is organized as follows: Section 2 discusses some necessary facts about lines in 3-space, and presents the concept of LN-surfaces. Section 3 discusses the different cases of surfaces and gives explicit parameterizations. Some examples illustrate the method. In Section 4 we conclude the article and give hints to possible applications.

2. GEOMETRIC BACKGROUND

Points in $\mathbb{R}^n$ are represented by their inhomogeneous coordinate vectors $\mathbf{x} = (x_1, \dots, x_n)$. The projective closure of $\mathbb{R}^n$ is denoted by $\mathbb{P}^n$. Points in $\mathbb{P}^n$ are identified with their *homogeneous coordinate vectors*

$$\mathbf{y}\mathbb{R} = (y_0, y_1, \dots, y_n)\mathbb{R}, \quad \text{with} \quad \mathbf{y} \neq \mathbf{o}. \tag{1}$$

Let $\omega : y_0 = 0$ be the hyperplane at infinity in $\mathbb{P}^n$. The interchange between homogeneous and Cartesian coordinates for points in $\mathbb{R}^n$ is realized by

$$x_1 = \frac{y_1}{y_0}, \; x_2 = \frac{y_2}{y_0}, \dots, \; x_n = \frac{y_n}{y_0}. \tag{2}$$

ISSAC'08, July 20–23, 2008, Hagenberg, Austria.

Moreover, let $\mathbb{P}^{n\star}$ be the dual projective space, whose points are identified with the hyperplanes in $\mathbb{P}^n$. A hyperplane $E : e_0y_0 + e_1y_1 + \ldots + e_ny_n = 0$ in $\mathbb{P}^n$ is identified with the homogeneous coordinate vector $\mathbf{e}\mathbb{R} = (e_0, \ldots, e_n)\mathbb{R}$.

2.1 LN-curves and LN-surfaces

A rational curve $C \subset \mathbb{R}^2$ is called an LN-curve if its tangent lines admit the representation $T(u) : x_1u + x_2 = f(u)$, where $f(u)$ is a rational function. This implies that the curve C possesses the *linear normal vector field* $\mathbf{n}(u) = (u, 1)$. A parameterization $\mathbf{c}(u)$ of C can be computed by intersecting $T \cap \dot{T}$. This leads to

$$\mathbf{c}(u) = (\dot{f}, f - u\dot{f}), \text{ with } \dot{f} = df/du. \tag{3}$$

Analogously LN-surfaces $F \subset \mathbb{R}^3$ are characterized as rational surfaces possessing a linear normal vector field $\mathbf{n}$. By excluding cylinders we can assume that $\mathbf{n} = (u, v, 1)$. The tangent planes $T(u, v)$ of F admit the representation

$$T : x_1u + x_2v + x_3 = f(u, v), \tag{4}$$

where $f(u, v)$ is a bivariate rational function. A parameterization $\mathbf{f}(u, v)$ of F can be computed as intersection $T \cap T_u \cap T_v$. This leads to

$$\mathbf{f}(u, v) = (f_u, f_v, f - uf_u - vf_v), \tag{5}$$

where $f_u = df/du$ and $f_v = df/dv$. These surfaces first occurred in computer aided geometric design in [2]. They have very special properties concerning envelope computation with respect to two-parameter translational motions, see [11]. LN-surfaces F in $\mathbb{R}^3$ are also characterized by the fact that for any plane ε in $\mathbb{R}^3$ the surface parameters of F can be expressed in terms of rational functions of the coefficients of ε. This implies that for all planes ε there exists a unique tangent plane T parallel to ε with a unique point of contact $\mathbf{f} \in F$.

The generalization of this concept to LN-hypersurfaces F in $\mathbb{R}^4$ is straightforward. Prescribing the tangent hyperplanes by

$$T(u, v, w) : x_1u + x_2v + x_3w + x_4 = f(u, v, w), \tag{6}$$

F is parameterized by $\mathbf{f} = (f_u, f_v, f_w, f - uf_u - vf_v - wf_w)$.

2.2 Basic facts on surfaces in 4-space

Let $\Phi \subset \mathbb{R}^4$ be a two-dimensional surface, and let $\mathbf{p} : (u, v) \in \mathbb{R}^2 \to \mathbb{R}^4$ be a parameterization of Φ. At a regular surface point $\mathbf{p}(u, v)$ the partial derivative vectors $\mathbf{p}_u$ and $\mathbf{p}_v$ determine the tangent plane T at $\mathbf{p}$. Considering $\mathbb{R}^4$ as Euclidean space with the canonical scalar product, there exist two linearly independent vectors $\mathbf{e}(u, v)$ and $\mathbf{f}(u, v)$ which determine the normal plane N at $\mathbf{p}$. These vectors satisfy the relations $\mathbf{e}^T\mathbf{p}_u = \mathbf{e}^T\mathbf{p}_v = 0$ and $\mathbf{f}^T\mathbf{p}_u = \mathbf{f}^T\mathbf{p}_v = 0$. The tangent plane T at $\mathbf{p}$ can be considered as intersection of the 3-spaces $(\mathbf{x} - \mathbf{p})^T\mathbf{e} = 0$ and $(\mathbf{x} - \mathbf{p})^T\mathbf{f} = 0$.

When studying rational surfaces Φ in $\mathbb{R}^4$ it is advantageous to consider the projective extension $\mathbb{P}^4$ of $\mathbb{R}^4$. Let $\omega = \mathbb{P}^4 \setminus \mathbb{R}^4$ be the ideal 3-space. The ideal lines $g = T \cap \omega$ of the tangent planes T of Φ form a rational two-parameter family of lines $\mathcal{G}$. Likewise the ideal lines $h = N \cap \omega$ of the normal planes T of Φ form a rational two-parameter family of lines $\mathcal{H}$. The rational surfaces which will be constructed in Sect. 3 are characterized by special properties of the families $\mathcal{G}$ and $\mathcal{H}$ in ω. Therefore we point to some facts from the geometry of lines in $\mathbb{P}^3$. For more details on this topic we refer to [9].

2.3 Lines in projective three-space

In order to introduce coordinates for lines in $\mathbb{P}^3$, let a line g be spanned by two different points $P = \mathbf{p}\mathbb{R}$ and $Q = \mathbf{q}\mathbb{R}$, with $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^4$. The *Plücker coordinates* $\mathbf{G} = (g_1, \ldots, g_6)$ of g are defined by

$$\mathbf{G} = \begin{pmatrix} p_0q_1 - p_1q_0, p_0q_2 - p_2q_0, p_0q_3 - p_3q_0, \\ p_2q_3 - p_3q_2, p_3q_1 - p_1q_3, p_1q_2 - p_2q_1 \end{pmatrix}. \tag{7}$$

The coordinates g_i are homogeneous and independent of the choice of the points P and Q. Thus, they can be interpreted as the coordinates of points $\mathbf{G}\mathbb{R} = (g_1, \ldots, g_6)\mathbb{R}$ in $\mathbb{P}^5$. The coordinates g_i are not independent but satisfy the *Plücker relation*

$$g_1g_4 + g_2g_5 + g_3g_6 = 0. \tag{8}$$

A line g in $\mathbb{P}^3$ can also be considered as intersection of two planes ε and φ. Let these planes be given by their homogeneous coordinate vectors $\mathbf{e} = (e_0, e_1, e_2, e_3)\mathbb{R}$ and $\mathbf{f} = (f_0, f_1, f_2, f_3)\mathbb{R}$. The Plücker coordinates $\mathbf{G} = (g_1, \ldots, g_6)$ of $g = \varepsilon \cap \varphi$ are computed by

$$\mathbf{G} = \begin{pmatrix} e_2f_3 - e_3f_2, e_3f_1 - e_1f_3, e_1f_2 - e_2f_1, \\ e_0f_1 - e_1f_0, e_0f_2 - e_2f_0, e_0f_3 - e_3f_0 \end{pmatrix}. \tag{9}$$

2.4 Special two-parameter families of lines

We study rational two-parameter families of lines $\mathcal{G}$ with the property that for almost all planes ε in $\mathbb{P}^3$ there exists a unique line $g \in \mathcal{G}$ with $g \subset \varepsilon$. Applying a duality or polarity δ in $\mathbb{P}^3$, the family of lines $\mathcal{G}$ is mapped to a family of lines $\mathcal{H} = \delta(\mathcal{G})$. The family $\mathcal{H}$ has the property that for almost all points $X \in \mathbb{P}^3$ there exists a unique line $h \in \mathcal{H}$ with $X \in h$. Algebraic families of lines of this type have an *exceptional set*, which means that there exists at most a one-parameter family of points where the line $h \in \mathcal{H}$ is not unique.

Since it is more intuitive in some cases we describe rational two-parameter families of lines $\mathcal{H}$ sending a unique line through a generic point. These families are called *congruences of lines* of degree one and class n, denoted as $(1, n)$-congruences. The degree denotes the number of lines passing through a generic point, and the class denotes the number of lines lying in a generic plane.

It is a result of classical algebraic line geometry [10], that, besides the star of lines, there exist the following types:

- Chordal variety of a spatial cubic: $(1, 3)$-congruence.
- $(1, n)$-congruences of the first kind: there exist two different singular curves.
- $(1, n)$-congruences of the second kind: there exists only one singular curve which is a line.

This classification holds for families of lines in complex projective space $\mathbb{P}^3$. Since we have to focus on families of real lines, we describe seven types of families of lines. Besides the star of lines (type 6) and the chordal variety (type 5) there exist two linear line congruences (type 1 and type 3) with different singular curves and one linear line congruence (type 2) with only one singular curve for $n = 1$. For types 1–3 the families $\mathcal{G}$ and $\mathcal{H}$ are of the same type. Type 4 describes $(1, n)$-congruences of the first kind are given in type 4. Analogously, type 7 deals with $(1, n)$-congruences of the second kind.

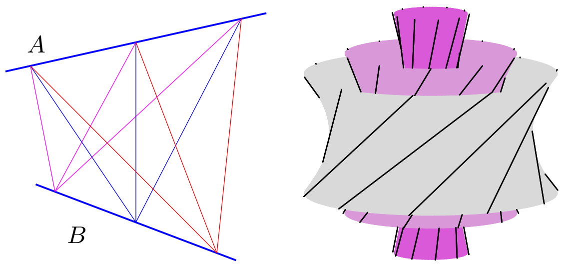

Figure 1: Line congruences of types 1 and 3.

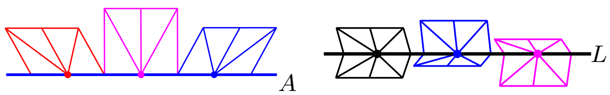

Figure 2: Line congruences of types 2 and 7.

Type 1 – hyperbolic linear line congruence: This family of lines in $\mathbb{P}^3$ consists of those lines intersecting two real skew lines A and B which are called the axes of the congruence. The exceptional set consists of A and B, considered as point set for $\mathcal{H}$ and as pencils of planes for $\mathcal{G}$.

Type 2 – parabolic linear line congruence: It consists of a one-parameter family of pencils of lines in planes through the *axis* A and with vertices on A. The correspondence between the carrier planes and the vertices of the pencils is a projective mapping. The exceptional set consists of the axis A, considered as point set for $\mathcal{H}$ and as pencil of planes for $\mathcal{G}$.

Type 3 – elliptic linear line congruence: This is a family of real lines in $\mathbb{P}^3$ which intersect a pair of skew and conjugate complex lines A and $\overline{A}$. Some authors use the notation *spread* for this family. The exceptional set does not contain real points or planes.

Type 4 – $(1, n)$-congruence of the first kind: Let C be an algebraic space curve of degree n and let L be a line intersecting C in $n-1$ points. The family $\mathcal{H}$ comprises the lines intersecting both C and L. For any generic point $X \notin C, L$ there exists a plane $X \vee L$ intersecting C in a further point Y and XY is the unique line of the family passing through X. The exceptional set consists of C and L.

Type 5 – chordal variety: This family $\mathcal{H}$ of lines consists of the chords of a spatial cubic C in $\mathbb{P}^3$. It contains also tangent lines of C and lines connecting two conjugate complex points of C. The points of the cubic C form the exceptional set.

Type 6 – star of lines: This family $\mathcal{H}$ consists of the lines through a fixed point P. It sends a unique line through any point $X \neq P$, and P is the exceptional set. The dual family $\mathcal{G}$ is called *ruled plane* and is formed by all lines lying in a fixed plane π, which is the exceptional set.

Type 7 – $(1, n)$-congruence of the second kind: Let L be a line. There exists a rational correspondence between the points $X \in L$ and the planes ε through L in a way that each point X corresponds to n planes but each plane $\varepsilon \supset L$ corresponds only to one single point. The family consists of pencils of lines with vertices $X \in L$ which lie in planes $\varepsilon \supset L$. The line L is the exceptional set of this family of lines.

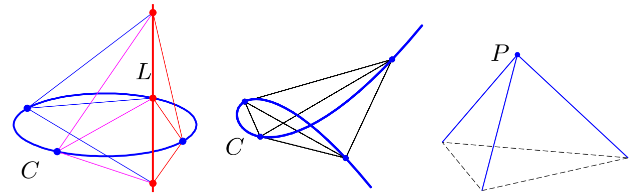

Figure 3: Line congruences of types 4,5, and 6.

For the types 4 and 7 it is not possible to give a complete list. Examples for the families of type 4 for $n = 2$ and $n = 3$ are given in Sect. 3.4. An example for a family of type 7 for $n = 2$ is given in Sect. 3.7.

3. CONSTRUCTION OF THE SURFACES

We present a construction for rational two-dimensional surfaces Φ in $\mathbb{R}^4$ which generalizes LN-surfaces. Let $\mathbf{p}(u, v)$ be a rational parameterization of Φ, and let $E : e_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 = 0$ be a 3-space in R^4.

Definition 1. A rational two-dimensional surface Φ in $\mathbb{R}^4$ is called *generalized LN-surface* if for all 3-spaces E the surface parameters u and v can be expressed in terms of rational functions depending on the coefficients e_i of E.

This implies that for all 3-spaces E there exists a unique tangent plane T of Φ which is parallel to E. Although this definition characterizes an affine-invariant class of surfaces, it is convenient to introduce the orthogonality in $\mathbb{R}^4$ determined by the canonical scalar product $\mathbf{x}^T\mathbf{y} = \sum_i x_i y_i$.

Let $\mathbb{P}^4$ be the projective extension of $\mathbb{R}^4$, and let $\omega = \mathbb{P}^4 \setminus \mathbb{R}^4$ be the ideal hyperplane (at infinity). The ideal lines $g = T \cap \omega$ of the tangent planes T of Φ form the family $\mathcal{G}$. Let $\varepsilon = E \cap \omega$ be the ideal plane of a 3-space E. According to Def. 1, these rational surfaces Φ are characterized by the fact that all planes $\varepsilon \in \omega$ carry a unique ideal line $g \in \mathcal{G}$. A duality δ in ω maps planes ε to points X and the family $\mathcal{G}$ to a family of lines $\mathcal{H} = \delta(\mathcal{G})$. If any plane ε carries a unique line $g \in \mathcal{G}$ then the family $\mathcal{H}$ sends a unique line $h \in \mathcal{H}$ through any point X.

Without loss of generality we may choose δ as polarity with respect to the quadric $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0$. This implies that the canonical scalar product in $\mathbb{R}^4$ is induced by δ. Therefore $\mathcal{H}$ is considered as the family of ideal lines $h = N \cap \omega$ of the normal planes N of Φ. The construction of the surfaces uses the classification of $(1, n)$-congruences from [10]. For each type of congruence of degree one we obtain an affine-invariant family of generalized LN-surfaces. Besides the surfaces from Sect. 3.6 we restrict our interest to surfaces Φ which span $\mathbb{R}^4$.

The proposed construction interprets a surface Φ in $\mathbb{R}^4$ as envelope of its tangent planes. Each plane $T(u, v)$ is considered as intersection of the 3-spaces

$$\begin{aligned} E(u,v): \quad \mathbf{e}(u,v)^T\mathbf{x} &= a(u,v), \\ F(u,v): \quad \mathbf{f}(u,v)^T\mathbf{x} &= b(u,v), \end{aligned} \tag{10}$$

where $\mathbf{e}(u,v)$ and $\mathbf{f}(u,v)$ can be considered as normal vector fields of Φ and $a(u,v)$ and $b(u,v)$ are rational functions. The ideal lines g of T are represented as intersection lines $g = \varepsilon \cap \varphi$ of the ideal planes $\varepsilon = E \cap \omega$ and $\varphi = F \cap \omega$ of E and F. Using $\mathbf{x}$ as homogeneous coordinates in ω, these planes are given by the equations $\varepsilon : \mathbf{e}^T\mathbf{x} = 0$ and $\varphi : \mathbf{f}^T\mathbf{x} = 0$. The question arises, which conditions have to be fulfilled by the functions $\mathbf{e}, \mathbf{f}$ and a, b such that the family of planes $T(u,v)$ possesses an envelope surface?

Corollary 1. *Let $T = E \cap F$ be a rational two-parameter family of planes in $\mathbb{R}^4$, where E and F are rational two-parameter families of hyperplanes. The planes T possess a rational envelope surface Φ if and only if the following system of linear equations has a solution,*

$$\begin{array}{llll} E: & \mathbf{e}^T\mathbf{x} = a, & F: & \mathbf{f}^T\mathbf{x} = b, \\ E_u: & \mathbf{e}_u^T\mathbf{x} = a_u, & F_u: & \mathbf{f}_u^T\mathbf{x} = b_u, \\ E_v: & \mathbf{e}_v^T\mathbf{x} = a_v, & F_v: & \mathbf{f}_v^T\mathbf{x} = b_v. \end{array} \tag{11}$$

Proof. Let $\mathbf{p}(u,v)$ be a solution of the system (11). We have to prove that $T = E \cap F$ is the tangent plane at a regular point $\mathbf{p}(u,v)$. Since T is parameterized by $\mathbf{p} + s\mathbf{p}_u + t\mathbf{p}_v$, this is equivalent to verify the relations

$$\mathbf{e}^T\mathbf{p}_u = 0, \mathbf{e}^T\mathbf{p}_v = 0 \text{ and } \mathbf{f}^T\mathbf{p}_u = 0, \mathbf{f}^T\mathbf{p}_v = 0.$$

Differentiating $\mathbf{e}^T\mathbf{p} = a$ with respect to u and v and taking $\mathbf{e}_u^T\mathbf{p} = a_u$, and $\mathbf{e}_v^T\mathbf{p} = a_v$ into account, leads to $\mathbf{e}^T\mathbf{p}_u = 0$ and $\mathbf{e}^T\mathbf{p}_v = 0$. Analogously we proceed for $\mathbf{f}$, which concludes the proof. $\square$

If the rank of the coefficient matrix of the system (11) is less than four in $\mathbb{R}^2$, the solution may degenerate. Since the normal forms of vectors $\mathbf{e}$ and $\mathbf{f}$ we use later always lead to matrices of rank four, we do not consider degenerate cases here.

Generalized LN-surfaces Φ according to Def. 1 can be characterized equivalently in the following way. Let $W : \mathbf{w}^T\mathbf{x} = c$ be an arbitrary 3-space in $\mathbb{R}^4$. The surface Φ is a generalized LN-surface if and only if it possesses a parameterization $\mathbf{p}(u,v)$ in a way that the equations

$$\mathbf{w}^T\mathbf{p}_u(u,v) = 0 \text{ and } \mathbf{w}^T\mathbf{p}_v(u,v) = 0 \tag{12}$$

possess rational solutions $u = \alpha(\mathbf{w})$, $v = \beta(\mathbf{w})$ depending on the coordinates w_i of $\mathbf{w} \in \mathbb{R}^4$.

Remark 1. From the parameterizations $\mathbf{p}(u,v)$ of the seven types of generalized LN-surfaces Φ in $\mathbb{R}^4$ discussed later one arrives at systems of polynomial equations (12). Without taking into account the geometric generation of these surfaces, one would have to use Gröbner basis or other elimination techniques to solve these equations for u and v. The geometric generation (11), however, induces a preferable computation, because we already know that $\mathbf{e}(u,v)$ and $\mathbf{f}(u,v)$ are normal vector fields of Φ. This allows to compute two vector fields $\mathbf{s}(u,v)$ and $\mathbf{t}(u,v)$ spanning the tangent spaces of Φ with methods from linear algebra. Substituting $\mathbf{p}_u$ and $\mathbf{p}_v$ by $\mathbf{s}$ and $\mathbf{t}$, equations (12) simplify. Moreover, the construction itself proves the rationality of the solutions for u and v.

Each of the types of rational families of lines from section 2.4 leads to a family of generalized LN-surfaces Φ in $\mathbb{R}^4$, according to Def. 1. In most cases these surfaces are studied by using special normal forms for functions $\mathbf{e}$ and $\mathbf{f}$, and conditions to the functions a and b for the existence of envelope surfaces Φ are determined. This is no restriction since the construction of the surfaces is invariant with respect to affine transformations in $\mathbb{R}^4$. In most cases these surfaces admit simple geometric generations and relations to LN-curves and LN-surfaces will appear.

3.1 Surfaces of type 1

If $\mathcal{H}$ is a hyperbolic linear line congruence, the family $\mathcal{G}$ is of the same type. The Plücker coordinates of the axes A and B of $\mathcal{G}$ are chosen as $\mathbf{A} = (0,0,0,0,1,0)$ and $\mathbf{B} = (0,1,0,0,0,0)$. The two pencils of planes $\varepsilon(u)$ and $\varphi(v)$ passing through A and B are parameterized by

$$\mathbf{e}(u) = (1,0,u,0) \text{ and } \mathbf{f}(v) = (0,1,0,v).$$

The tangent planes T of Φ have ideal lines $g = \varepsilon \cap \varphi$, and its Plücker coordinates are $\mathbf{G} = (uv, 0, -u, 1, 0, v)$. Since $\mathbf{e}(u)$ and $\mathbf{f}(v)$ are univariate polynomials, $a(u)$ and $b(v)$ have to be chosen as univariate rational functions. They satisfy

$$a_v = 0 \text{ and } b_u = 0. \tag{13}$$

Vectors $\mathbf{e}$ and $\mathbf{f}$ define two independent normal vector fields of Φ. A rational parameterization of Φ is obtained as solution of the system (11), and reads

$$\mathbf{p}(u,v) = (a - ua_u, b - vb_v, a_u, b_v). \tag{14}$$

Thus a surface Φ of type 1 is a translational surface $\mathbf{p}(u,v) = \mathbf{c}(u) + \mathbf{d}(v)$ with profile curves C and D,

$$\begin{array}{ll} \mathbf{c}(u) & = (a - ua_u, 0, a_u, 0), \\ \mathbf{d}(v) & = (0, b - vb_v, 0, b_v). \end{array} \tag{15}$$

Comparison with (3) shows that C and D are LN-curves. The tangent planes T of Φ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (-u, 0, 1, 0)$ and $\mathbf{t} = (0, -v, 0, 1)$. The partial derivatives can be expressed by

$$\mathbf{p}_u = a_{uu}\mathbf{s} \text{ and } \mathbf{p}_v = b_{vv}\mathbf{t}.$$

According to (12) the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in the rational expressions

$$u = \frac{w_3}{w_1} \text{ and } v = \frac{w_4}{w_2}. \tag{16}$$

Theorem 1. *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 1 is a translational surface $\mathbf{p}(u,v) = \mathbf{c}(u) + \mathbf{d}(v)$ with planar LN-curves as profile curves. Conversely, any translational surface Φ which spans $\mathbb{R}^4$ and whose profile curves are LN-curves is a surface of type 1.*

Proof. We only have to show that a translational surface F which spans $\mathbb{R}^4$ obtained by translating LN-curves C and D is equivalent to (14). Since C and D shall span $\mathbb{R}^4$, we may choose an affine coordinate system in a way that C and D are parameterized by (15). Since F is parameterized by $\mathbf{f}(u,v) = \mathbf{c}(u) + \mathbf{d}(v)$, the statement holds. $\square$

Example 1. Choosing quadratic polynomials $a(u) = 1/2u^2$ and $b(v) = 1/2v^2$, the surface of type 1 is a quadratically parameterized surface $\mathbf{p} = (-1/2u^2, -1/2v^2, u, v)$, see [8]. Fig. 4 illustrates the projection $\mathbf{p} = 1/2(-u^2, -v^2, u+v)$ in $\mathbb{R}^3$.

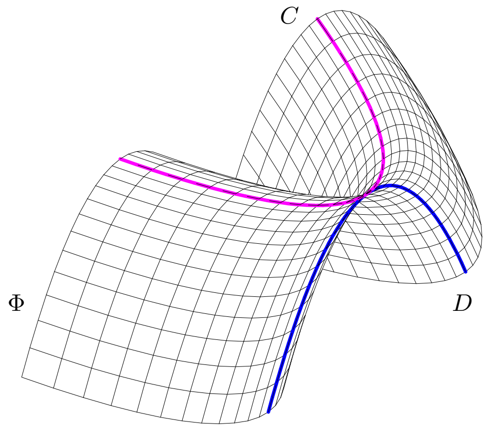

Figure 4: Projection of a surface of type 1 into $\mathbb{R}^3$.

3.2 Surfaces of type 2

If $\mathcal{H}$ is a parabolic linear line congruence, the family $\mathcal{G}$ is of the same type. The lines $g(u,v)$ of $\mathcal{G}$ are the intersection lines of the pencil of planes $\varepsilon(u)$ through the axis A of $\mathcal{G}$ and an appropriate star of planes $\varphi(u,v)$ passing through a vertex $\notin A$. We choose

$$\mathbf{e}(u) = (1,0,u,0) \text{ and } \mathbf{f}(u,v) = (0,-1,v,u).$$

The family $\mathcal{G}$ consists of the pencils of lines with vertices $(0,1,0,u)\mathbb{R}$ lying in planes $\varepsilon(u)$. The Plücker coordinates of the lines $g(u,v)$ are $\mathbf{G} = (u^2, 0, u, -1, v, u)$. Since $\mathbf{e}_v = (0,0,0,0)$ and $\mathbf{e}_u = \mathbf{f}_v$ holds, the rational functions $a(u)$ and $b(u,v)$ have to satisfy the relations

$$a_v = 0 \text{ and } a_u - b_v = 0. \tag{17}$$

This implies $b(u,v) = va_u - \lambda(u)$. Solving (11), the surface Φ is parameterized by

$$\mathbf{p}(u,v) = (a - ua_u, ub_u + \lambda, a_u, b_u).$$

The tangent planes T of Φ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (-u, v, 1, 0)$ and $\mathbf{t} = (0, u, 0, 1)$. The partial derivatives can be expressed by

$$\mathbf{p}_v = a_{uu}\mathbf{t} \text{ and } \mathbf{p}_u = a_{uu}\mathbf{s} - (\lambda_{uu} - va_{uuu})\mathbf{t}.$$

According to (12) the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in the rational expressions

$$u = -\frac{w_4}{w_2} \text{ and } v = -\frac{w_1 w_4 + w_2 w_3}{w_2^2}. \tag{18}$$

Since $b_u = va_{uu} - \lambda_u$ holds, $\mathbf{p}(u,v)$ is linear in v and Φ is a ruled surface. Letting $\tilde{v} = b_u$, one obtains the ruled surface parameterization

$$\begin{aligned} \mathbf{p}(u,v) &= \mathbf{c}(u) + \tilde{v}\mathbf{d}(u) \\ &= (a - ua_u, \lambda, a_u, 0) + \tilde{v}(0, u, 0, 1). \end{aligned} \tag{19}$$

The directrix curve $\mathbf{c}(u)$ is a rational curve on a cylinder over the LN-curve $(a - ua_u, 0, a_u)$, and $\mathbf{d}(u)$ is a linear direction vector field along $\mathbf{c}$. The reparameterization $\tilde{v} = b_u$ has influence on the parameterization of $\mathcal{G}$. However, for the geometric properties of surfaces of type 2 this is not relevant.

THEOREM 2. *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 2 is a ruled surface with a rational curve C on a cylinder over a LN-curve as directrix curve and a linear direction vector field. Conversely, any ruled surface which spans $\mathbb{R}^4$ and possesses these properties is a surface of type 2.*

PROOF. To prove the converse statement one only has to choose an appropriate affine coordinate system and obtains the parameterization (19). □

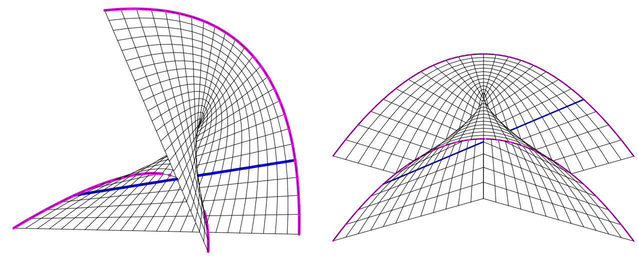

Figure 5: Cayley's surface(left) and Whitney's umbrella(right)

Example 2. For $a(u) = 1/2u^2$ we obtain the surface $\mathbf{f}(u,v) = (-1/2u^2, uv, u, v)$. It can also be represented as quadratically parameterized surface, see [8]. The projection onto $x_4 = 0$ in direction of $(1,0,0,-1)$ is recognized as *Cayley's-surface* with parameterization $\widetilde{\mathbf{f}}(u,v) = (-1/2u^2 + v, uv, u)$. The orthogonal projection onto $x_3 = 0$ is known as *Whitney's umbrella* or *Plücker's conoid* with parameterization $\widetilde{\mathbf{f}}(u,v) = (-1/2u^2, uv, v)$.

3.3 Surfaces of type 3

If $\mathcal{H}$ is an elliptic linear line congruence, the family $\mathcal{G}$ is of the same type. In order to describe $\mathcal{G}$, let α be a projective mapping between two stars of planes $\varepsilon(u,v)$ and $\varphi(u,v)$. Their vertices are chosen as $V_e = (0,1,0,0)\mathbb{R}$ and $V_f = (1,0,0,0)\mathbb{R}$, and the planes ε and φ can be determined by

$$\mathbf{e}(u,v) = (1,0,-u,v) \text{ and } \mathbf{f}(u,v) = (0,-1,v,u),$$

Their intersection lines $g(u,v) \in \mathcal{G}$ are given by the Plücker coordinates $\mathbf{G} = (u^2 + v^2, v, u, 1, -v, -u)$. The projective mapping $\alpha : \varepsilon \mapsto \varphi$ maps the pencil of planes through $V_e V_f$ onto itself, $\eta(t) = (0,0,t,1)\mathbb{R} \mapsto \alpha(\eta(t)) = (0,0,1,-t)\mathbb{R}$. Since α restricted to the pencil $\eta(t)$ possesses two conjugate complex fixed planes $(0,0,i,1)\mathbb{R}$ and $(0,0,1,-i)$, the family $\mathcal{G}$ of lines $g = \varepsilon \cap \varphi$ is an elliptic linear line congruence.

Because of $\mathbf{e}_u = -\mathbf{f}_v$ and $\mathbf{e}_v = \mathbf{f}_u$, a parameterization $\mathbf{p}(u,v)$ of a surface Φ of type 3 is obtained as solution of (11) for any choice of rational functions $a(u,v)$ and $b(u,v)$ which satisfy

$$a_u = -b_v \text{ and } a_v = b_u. \tag{20}$$

Thus a and b are harmonic conjugate and b is the real part and a is the imaginary part of a univariate polynomial or rational function in the complex variable $z = u + iv$. A rational parameterization of Φ reads

$$\mathbf{p}(u,v) = (a - ua_u - va_v, -b - va_u + ua_v, -a_u, a_v). \tag{21}$$

We show that a surface Φ of type 3 is a translational surface with two conjugate complex profile curves C and $\overline{C}$ with parameterizations $\mathbf{c}(z)$ and $\overline{\mathbf{c}(z)}$, respectively. Let $f(z)$ be a rational function in the complex variable z with imaginary part $a(u,v)$ and real part $b(u,v)$. Thus $f = b + ia$. Differentiating f with respect to z and $\bar{z}$ gives

$$\frac{df}{dz} = f_z = a_v + ia_u \text{ and } \frac{df}{d\bar{z}} = 0.$$

It is not difficult to verify that Φ can be represented by

$$\begin{array}{l}\mathbf{p}(u,v) = \frac{1}{2}(\mathbf{c}(z) + \overline{\mathbf{c}(z)}), \text{ with} \\ \mathbf{c}(z) = (i(zf_z - f), (zf_z - f), if_z, f_z)\,,\end{array} \tag{22}$$

where $\mathbf{c}(z)$ is a planar curve. Its carrier plane is given by $ix_1 + x_2 = ix_3 + x_4 = 0$. Since the tangent lines of C have direction vectors $d\mathbf{c}/dz = f_{zz}(iz, z, i, 1)$, the curves C and $\overline{C}$ are a pair of planar conjugate complex LN-curves. The tangent planes T of Φ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (u,v,1,0)$ and $\mathbf{t} = (-v,u,0,1)$. The partial derivatives can be expressed by

$$\mathbf{p}_u = -a_{uu}\mathbf{s} + b_{uu}\mathbf{t} \text{ and } \mathbf{p}_v = b_{vv}\mathbf{s} + a_{vv}\mathbf{t}.$$

According to (12) the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in the system of linear equations for u and v,

$$\begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -w_3 \\ -w_4 \end{pmatrix}. \tag{23}$$

THEOREM 3. *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 3 is a translational surface $\mathbf{p}(z) = 1/2(\mathbf{c}(z) + \overline{\mathbf{c}(z)})$ with a pair of planar conjugate complex LN-curves as profile curves. Conversely, any translational surface Φ whose profile curves are a pair of conjugate complex LN-curves which span $\mathbb{R}^4$ is a surface of type 3.*

PROOF. We only have to show that a translational surface F obtained by translating LN-curves C and $\overline{C}$ is equivalent to (21). Since we require that C and $\overline{C}$ shall span $\mathbb{R}^4$, we can choose an affine coordinate system in a way that C is parameterized by $\mathbf{c}(z)$ from (22). Since F is parameterized by $\mathbf{p}(z) = 1/2(\mathbf{c}(z) + \overline{\mathbf{c}(z)})$, the theorem holds. □

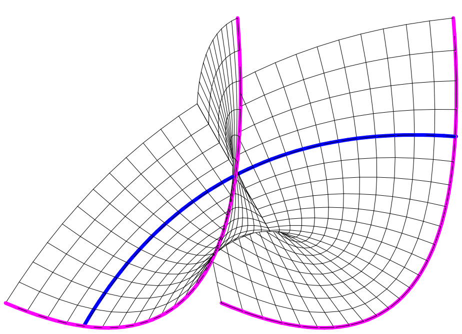

Figure 6: Projection of a surface of type 3.

Example 3. For the choice $a = 1/2(v^2 - u^2)$ and $b = uv$ one obtains the quadratically parameterized surface $\mathbf{p} = (1/2(u^2 - v^2), uv, u, v)$. A projection of this surface onto $x_4 = 0$ is displayed in Fig. 6.

3.4 Surfaces of type 4

Let C be a spatial algebraic curve of degree n in ω which has at least one chord L joining $n-1$ points of C. The number of points is counted algebraically. The family $\mathcal{H}$ of lines h meeting the curve C and the line L sends a unique line through a generic point $X \in \omega$.

Since there exist more than one family $\mathcal{H}$ for $n > 2$, we only give examples for $n = 2$ and $n = 3$:

- Let C be a conic and let L be a line meeting C at a single point. The family $\mathcal{H}$ consists of all lines meeting both C and L. The polarity maps C to a quadratic cone D, and L to a tangent line M of D.

- Let C be a cubic and let L be a chord of C. The family $\mathcal{H}$ consists of all lines meeting both C and L. The chord L of C can be replaced by a tangent line of C. The polarity maps C to a developable surface D of class three, and L to a line M carrying two tangent planes of D. These two tangent planes are not necessarily real.

The construction of a surface of type 4 is illustrated by the example for $n = 2$.

The planes $\varepsilon(u)$ and $\varphi(v)$ can be represented by

$$\mathbf{e}(u) = (-1,0,0,u) \text{ and } \mathbf{f}(v) = (0,u,-v,v^2).$$

The Plücker coordinates of lines $g(u,v) = \varepsilon(u) \cap \varphi(v)$ are $\mathbf{G} = (uv, u^2, 0, -u, v, -v^2)$. The rational functions a and b have to satisfy

$$a_v = 0 \text{ and } ub_u + vb_v - b - v^2 a_u = 0. \tag{24}$$

Choosing a as univariate rational function $a(u)$ and with $b(u,v)$ being of the form $b(u,v) = uh(u/v) + v^2/ua(u)$ a parameterization of this particular subclass of surfaces Φ of type 4 reads

$$\mathbf{p}(u,v) = (ua_u - a, b_u, 2va_u - b_v, a_u). \tag{25}$$

The tangent planes T of Φ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (u,0,v,1)$ and $\mathbf{t} = (0,v,u,0)$. The partial derivatives can be expressed by

$$\mathbf{p}_u = a_{uu}\mathbf{s} + \frac{b_{uu}}{v}\mathbf{t} \text{ and } \mathbf{p}_v = \frac{b_{uv}}{v}\mathbf{t}.$$

According to (12) the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in the rational expressions

$$u = \frac{w_2 w_4}{w_3^2 - w_1 w_2} \text{ and } v = \frac{-w_3 w_4}{w_3^2 - w_1 w_2}. \tag{26}$$

Example 4. For the choice $a = 1/2u^2$ and $b = 1/2uv^2$ one obtains the quadratically parameterized surface $\mathbf{p} = (1/2u^2, 1/2v^2, uv, u)$. A projection of this surface onto $x_1 = 0$ is the Plücker's conoid as displayed in Fig. 5. Another projection of this surface reads $(p_1 - p_2, p_3, p_4)$, which is displayed in Fig. 6.

3.5 Surfaces of type 5

Let $\mathcal{H}$ be the chordal variety of a spatial cubic C. Applying the polarity δ maps $\mathcal{H}$ to the axes variety $\mathcal{G}$ of a developable surface D of class 3. The family $\mathcal{G}$ consists of the real intersection lines of two tangent planes of D. These planes can also be conjugate complex. In addition the generating lines of D belong to $\mathcal{G}$.

Let $\varepsilon(t)$ be the real tangent planes of D. The intersection lines $g(u,v) = \varepsilon(u) \cap \varepsilon(v)$ form a subset of $\mathcal{G}$. This parameterization however has the disadvantage that each line g is obtained for two different parameter values (u_1, v_1) and (v_1, u_1). Consequently the uniqueness property does not hold for this representation. This can be avoided by using proper parameterizations of the axes variety $\mathcal{G}$.

The lines $g \in \mathcal{G}$ can be obtained by joining points $\mathbf{x}\mathbb{R} = (u, 0, v, 1)\mathbb{R}$ and $\mathbf{y}\mathbb{R} = (1, v, u, 0)\mathbb{R}$. Because of the linearity in u and v, the correspondence $\alpha : \mathbf{x}\mathbb{R} \mapsto \mathbf{y}\mathbb{R}$ is a projective mapping between the planes $y_1 = 0$ and $y_3 = 0$. The Plücker coordinates of lines g are $\mathbf{G} = (uv, u^2 - v, -1, -u, v, -v^2)$. In order to generate the surface Φ in $\mathbb{R}^4$ as envelope of its tangent planes, lines $g \in \mathcal{G}$ are considered as intersection of planes ε and φ. Their coordinate vectors are chosen as

$$\mathbf{e} = (u, 0, -1, -u^2 + v) \text{ and } \mathbf{f} = (-v, 1, 0, uv). \tag{27}$$

By investigating the equations (11), we realize that the system is solvable if and only if the functions $a(u,v)$ and $b(u,v)$ satisfy the conditions

$$b_u = v a_v \text{ and } b_v = -a_u - u a_v. \tag{28}$$

This system is integrable if the function $a(u,v)$ satisfies

$$a_{uu} = -u a_{uv} - 2a_v - v a_{vv}. \tag{29}$$

A solution for $b(u,v)$ follows by

$$b = \int v a_v du - \int \Big(\int (v a_{vv} + a_v) du + a_u + u a_v\Big) dv + C. \tag{30}$$

Reducing our interest to polynomial solutions, the integration of $b(u,v)$ is not problematic. The condition (29) for the function $a(u,v)$ can be satisfied with a polynomial ansatz. The corresponding parameterization of a surface Φ in $\mathbb{R}^4$ of type 5 is

$$\mathbf{p}(u,v) = (u a_v - b_v, b - v b_v, -a + b_u - u b_v, a_v)\,. \tag{31}$$

This representation depends on the choice of $\mathbf{e}$ and $\mathbf{f}$ which determine the parameterization of the axes variety $\mathcal{G}$. Different parameterizations typically lead to different representations for surfaces Φ of type 5. The tangent planes T of Φ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (u, 0, v, 1)$ and $\mathbf{t} = (1, v, u, 0)$. The partial derivatives can be expressed by

$$\mathbf{p}_u = a_{uv}\mathbf{s} - v a_{vv}\mathbf{t} \text{ and } \mathbf{p}_v = a_{vv}\mathbf{s} - b_{vv}\mathbf{t}.$$

According to (12) the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in the system of linear equations for u and v,

$$\begin{pmatrix} w_1 & w_3 \\ w_3 & w_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -w_4 \\ -w_1 \end{pmatrix}. \tag{32}$$

THEOREM 4. *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 5 admits explicit rational parameterizations. The tangent planes of Φ have linearly parameterized direction vectors $\mathbf{s} = (u, 0, v, 1)$ and $\mathbf{t} = (1, v, u, 0)$.*

Example 5. Performing a polynomial ansatz for $a(u,v)$ of degree three, a general solution of (28) for a and b depending on the coefficients c_i reads

$$\begin{aligned} a(u,v) &= c_4 + c_3 u - c_2 u^2 - 1/2 c_1 u^3 + c_2 v + c_1 uv, \\ b(u,v) &= c_2 uv + 1/2 c_1 u^2 v - c_3 v - 1/2 c_1 v^2. \end{aligned}$$

The coefficients $(c_1, \ldots, c_4) = (0, 0, 0, 1)$ give the polynomials $a(u,v) = -1/2u^3 + uv$ and $b(u,v) = 1/2u^2 v - 1/2v^2$. Finally the surface Φ of type 5 is a quadratically parameterized surface, parameterized by $\mathbf{p}(u,v) = (1/2u^2 + v, 1/2v^2, uv, u)$. Fig. 7 displays a projection onto $x_4 = 0$.

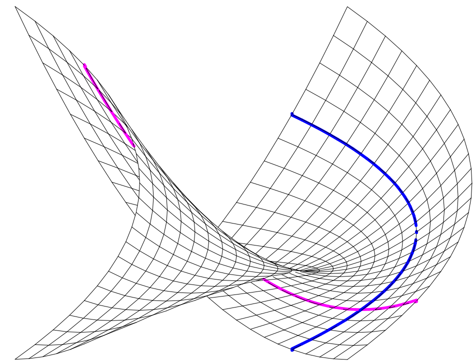

Figure 7: Projection of a surface of type 5.

3.6 Surfaces of type 6

If $\mathcal{H}$ is a star of lines, $\mathcal{G}$ is a ruled plane. The ideal lines $g(u,v)$ of the tangent planes $T(u,v)$ of a surface Φ of type 6 are obtained by intersecting the planes of a star $\varepsilon(u,v)$ with the fixed carrier plane φ of $\mathcal{G}$. We choose

$$\mathbf{e}(u,v) = (1, u, v, 0) \text{ and } \mathbf{f} = (0, 0, 0, 1)$$

as parameterizations of ε and φ, respectively and an arbitrary bivariate rational function $a(u,v)$ and $b = \text{const}$. Since $\mathcal{G}$ is a ruled plane, Φ is contained in a 3-space. Solving (11) a rational parameterization of a surface Φ of type 6 reads

$$\mathbf{p}(u,v) = (a - u a_u - v a_v, a_u, a_v, b)\,. \tag{33}$$

Comparison with section 2.1 shows that Φ is an LN-surface contained in $x_4 = b$. An example for $a(u,v) = u^3 + v^3$ and $b = 0$ is displayed in Fig. 8. The tangent planes T of Φ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (-u, 1, 0, 0)$ and $\mathbf{t} = (-v, 0, 1, 0)$. The partial derivatives can be expressed by

$$\mathbf{p}_u = a_{uu}\mathbf{s} + a_{uv}\mathbf{t} \text{ and } \mathbf{p}_v = a_{uv}\mathbf{s} + a_{vv}\mathbf{t}.$$

According to (12) the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in the rational expressions

$$u = \frac{w_2}{w_1} \text{ and } v = \frac{w_3}{w_1}. \tag{34}$$

THEOREM 5. *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 6 is an LN-surface in a hyperplane of $\mathbb{R}^4$.*

3.7 Surfaces of type 7

Let $\mathcal{H}$ be a $(1,n)$-congruence of the second kind. It possesses a singular line L and consists of pencils of lines with vertices $X \in L$ with carrier planes $\varepsilon \supset L$. Thereby each point $X \in L$ corresponds to n planes through L but each plane $\varepsilon \supset L$ corresponds only to one point $X \in L$. The case $n = 1$ is exactly type 2, and we give an example for $n = 2$. To represent planes ε and φ we choose

$$\mathbf{e}(u) = (1 - u^2, 0, 0, 2u) \text{ and } \mathbf{f}(u,v) = (0, 1, u, v).$$

The Plücker coordinates of lines $g = \varepsilon \cap \varphi$ are $\mathbf{G} = (-2u^2, 2u, 0, 1-u^2, u(1-u^2), v(1-u^2))$. Investigating the equations

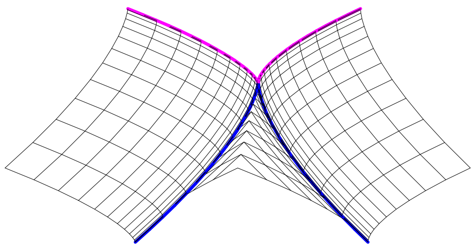

Figure 8: Surface of type 6.

(11), the functions $a(u, v)$ and $b(u, v)$ have to satisfy the conditions

$$a_v = 0 \text{ and } 2b_v(1 + u^2) = a_u(1 - u^2) + 2ua. \tag{35}$$

Thus the solvability of (11) requires a univariate rational function $a(u)$ and b is determined by

$$b(u, v) = \frac{v}{2(1 + u^2)}(a_u(1 - u^2) + 2ua) + \lambda(u).$$

A parameterization $\mathbf{p}$ of Φ is obtained by

$$\mathbf{p}(u, v) = \left(\frac{a - ua_u}{1 + u^2}, b - ub_u - vb_v, b_u, b_v \right). \tag{36}$$

The tangent planes T of Φ are spanned by vectors $\mathbf{s} = (0, -u, 1, 0)$ and $\mathbf{t} = (2u/(u^2 - 1), -v, 0, 1)$. The partial derivatives can be expressed by

$$\mathbf{p}_u = b_{uu}\mathbf{s} + b_{uv}\mathbf{t} \text{ and } \mathbf{p}_v = b_{uv}\mathbf{s}.$$

According to (12) the equations $\mathbf{w}^T\mathbf{s} = 0$ and $\mathbf{w}^T\mathbf{t} = 0$ result in the rational expressions

$$u = \frac{w_3}{w_2} \text{ and } v = \frac{2w_1w_2w_3 + w_4(w_3^2 - w_2^2)}{w_2(w_3^2 - w_2^2)}. \tag{37}$$

Since $b = vb_v + \lambda$ is linear in v, surfaces Φ are ruled surfaces. By letting $\tilde{v} = b_u$ we find the ruled surface parameterization

$$\mathbf{p}(u, v) = \left(\frac{a - ua_u}{1 + u^2}, \lambda, 0, \frac{a_u(1 - u^2) + 2ua}{2(1 + u^2)} \right) + \tilde{v}\,(0, -u, 1, 0).$$

Corollary 2. *A generalized LN-surface Φ in $\mathbb{R}^4$ of type 7 is a ruled surface whose generating lines are parallel to a fixed plane.*

Fig. 9 shows two different projections of the surface for $a = (1 + u^2)^2$ and $\lambda = 0$.

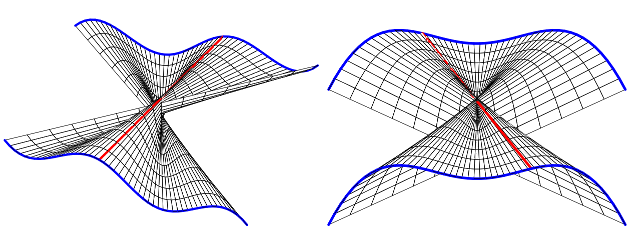

Figure 9: Two projections of a surface of type 7.

4. CONCLUSION

We have presented a class of rational surfaces Φ in $\mathbb{R}^4$ which satisfy the property that for all given vectors $\mathbf{w} \in \mathbb{R}^4$ the surface parameters u, v of Φ can be expressed by rational functions of the coefficients w_i of $\mathbf{w}$. Considering $\mathbb{R}^4$ as model space for the four-parameter family of spheres in $\mathbb{R}^3$, the presented surfaces correspond to two-parameter families of spheres whose envelope surfaces and their offsets admit rational parameterizations. These relations will be elaborated in a separate publication.

Acknowledgments

This work has been funded in parts by the Austrian Science Fund FWF within the research network S92.

5. REFERENCES

[1] Farouki, R.T., 2002. Pythagorean Hodograph curves, in: *Handbook of Computer Aided Geometric Design*, Farin, G., Hoschek, J., Kim, M.-S. eds.), Elsevier.

[2] Jüttler, B., 1998. Triangular Bézier surface patches with a linear normal vector field, in: *The Mathematics of Surfaces VIII*, Information Geometers, 431–446.

[3] Jüttler, B. and Sampoli, M.L., 2000. Hermite interpolation by piecewise polynomial surfaces with rational offsets, Comp. Aided Geom. Design **17**, 361–385.

[4] Kosinka, J. and Jüttler,B., 2007. MOS surfaces: Medial Surface Transforms with Rational Domain Boundaries, in: The Mathematics of Surfaces XII, R. Martin, M. Sabin, J. Winkler (eds.), Lecture Notes in Computer Science, Springer, 245–262.

[5] Krasauskas, R., 2006. Minimal rational parameterization of canal surfaces, Computing **79**, 281–290.

[6] Peternell, M. and Odehnal, B., 2008. Convolution surfaces of quadratic triangular Bézier surfaces, Comp. Aided Geom. Design **25**, 116-129.

[7] Peternell, M., Odehnal, B. and Sampoli, M.L. On quadratic two-parameter families of spheres and their envelope, Comp. Aided Geom. Design, in press.

[8] Peters, J., Reif, U., 1998. The 42 equivalence classes of quadratic surfaces in affine n-space, Comp. Aided Geom. Design **15**, 459–473.

[9] Pottmann, H. and Wallner, J., 2001. *Computational Line Geometry.* Springer-Verlag.

[10] Sturm, R., 1893. *Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung, II. Theil*, Teubner, Leibzig.

[11] Sampoli, M.L., Peternell, M. and Jüttler, B., 2006. Exact parameterization of convolution surfaces and rational surfaces with linear normals, *Computer Aided Geometric Design* **23**, 179–192.

[12] Zindler, K., 1921. Algebraische Liniengeometrie, in: Encykl. d. Math. Wiss. III, 2.2, 973–1228.