An Introduction to Line Geometry with Applications

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The paper presents a brief tutorial on classical line geometry and investigates new aspects of line geometry which arise in connection with a computational treatment. These mainly concern approximation and interpolation problems in the set of lines or line segments in Euclidean 3–space. In particular, we study approximation of data lines by, in a certain sense, 'linear' families of lines. These sets are for instance linear complexes and linear congruences. An application is the reconstruction of helical surfaces or surfaces of revolution from scattered data points. This is based on the fact that the normals of these surfaces lie in linear complexes; in particular, normals of surfaces of revolution intersect the axis of revolution.

Approximation with linear complexes or congruences is also useful to detect singular positions of serial or parallel robots. These are positions where the robot should be a rigid system but possesses an undesirable and unexpected instantaneous self motion.

Keywords: line geometry, reverse engineering, robot kinematics, ruled surface

Introduction and Motivation

Line geometry investigates the set of lines in 3–space. The ambient space can be a real projective, affine, Euclidean or a non–euclidean space. There is a vast literature on this branch of classical geometry including several monographs [6, 8, 10, 22, 34, 36].

Line geometry possesses a close relation to *spatial kinematics* [2, 14, 16, 34] and has therefore found applications in mechanism design and robot kinematics. Despite recent applications in robotics, a careful study of line geometry from the computational point of view has not been undertaken so far.

Let us look at an example to motivate the development of *computational line geometry*: A parallel manipulator has a singular position iff the axes of its hydraulic cylindrical legs lie in a linear complex, a special 3–parameter set of lines. In practice, several sources for errors (manufacturing, material properties, computing, ...) are hardly avoidable. Thus, the question is whether the lines near their realization on the objects are close — within some tolerance — to lines of a linear complex. This is an approximation or regression problem in line space.

In this paper we will survey central results of classical line geometry and also present new results on computational line geometry. The general concepts will be illustrated at hand of specific applications in areas such as robot kinematics and surface reconstruction. We also prepare a computational study of ruled surfaces, which appears in a separate paper of the same issue [23].

Line geometry and Plücker coordinates

Let us introduce our notation and get started with some essential fundamentals. First we work in real Euclidean space E^3. We use a Cartesian coordinate system and points are represented by their coordinate vectors $\mathbf{x}$. A straight line L can be determined by a point $\mathbf{p} \in L$ and a normalized direction vector

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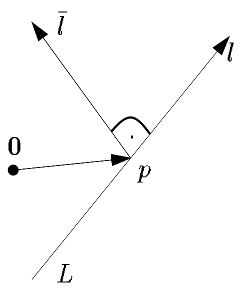

Figure 1: Direction and moment vector of a line

$\mathbf{l}$ of L, i.e. $||\mathbf{l}|| = 1$. To obtain coordinates for L one forms the moment vector

$$\bar{\mathbf{l}} := \mathbf{p} \times \mathbf{l} \tag{1}$$

with respect to the origin. If $\mathbf{p}$ is substituted by any point $\mathbf{q} = \mathbf{p} + \lambda\mathbf{l}$ on L, formula (1) says that $\bar{\mathbf{l}}$ is independent of the special choice of $\mathbf{p}$ on L. The 6 coordinates $(\mathbf{l}, \bar{\mathbf{l}})$ with

$$\mathbf{l} = (l_1, l_2, l_3), \text{ and } \bar{\mathbf{l}} = (l_4, l_5, l_6)$$

are called *normalized Plücker coordinates* of L. The set of lines in E^3 is a four-dimensional manifold and accordingly the six Plücker coordinates satisfy two relations. One is the normalization and the other one is, by (1), the so-called *Plücker relation*

$$\mathbf{l} \cdot \bar{\mathbf{l}} = 0, \tag{2}$$

which expresses the orthogonality of $\mathbf{l}$ and $\bar{\mathbf{l}}$.

Conversely, any 6–tupel $(\mathbf{l}, \bar{\mathbf{l}})$ with $||\mathbf{l}|| = 1$, which satisfies the Plücker relation $\mathbf{l} \cdot \bar{\mathbf{l}} = 0$ represents a line in E^3. Since we do not care about the orientation, $(\mathbf{l}, \bar{\mathbf{l}})$ and $(-\mathbf{l}, -\bar{\mathbf{l}})$ describe the same line L.

For some applications it is convenient to study lines in real projective space P^3, which can be considered as projective extension of E^3. This is the closure of E^3 by all points at infinity (intersections of parallel lines; for a simple introduction to concepts from projective geometry we refer the reader to Boehm and Prautzsch [1]). We use homogeneous Cartesian coordinate vectors $(x_0, \ldots, x_3) = \mathsf{x}$. These coordinates are only determined up to a common scalar multiple, which means that

$$\mathsf{x} \text{ and } \mathsf{x}\rho, \rho \in \mathbb{R} \setminus \{0\},$$

define the same point which shall be denoted by $\mathsf{x}\mathbb{R}$. A point at infinity is characterized by $x_0 = 0$ and points not at infinity possess the corresponding inhomogeneous Cartesian coordinates $(x_1/x_0, x_2/x_0, x_3/x_0)$. It is convenient to collect the projective homogeneous coordinates x in the following way

$$\mathsf{x} = (x_0, \mathbf{x}) \text{ with } x_0 \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3.$$

Let a plane in P^3 be given by a linear homogeneous equation

$$u_0 x_0 + u_1 x_2 + u_2 x_2 + u_3 x_3 = 0. \tag{3}$$

The vector $\mathsf{u} = (u_0, u_1, u_2, u_3)$ is called homogeneous plane coordinate vector of that plane. Since the coordinates u_i are also only determined up to a common scalar multiple we denote the plane by $\mathbb{R}\mathsf{u}$. The left and right scalar multiples in plane and point coordinates are just a convention to distinguish them. Analogously one uses the notation $\mathsf{u} = (u_0, \mathbf{u})$ with $u_0 \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^3$. All points at infinity form the plane at infinity (ideal plane) $x_0 = 0$, which is in our notation represented by the vector $(1, \mathbf{o})$. Any other plane possesses $\mathbf{u}$ as normal vector.

Now points and planes are represented in exactly the same way. Moreover, equation (3) can be seen as incidence relation between point $(x_0, \mathbf{x})\mathbb{R}$ and plane $\mathbb{R}(u_0, \mathbf{u})$ and it is symmetric. This is the basis of the principle of duality in projective geometry, which we will encounter later. Basically it says that exchanging the meaning of points and planes and keeping lines and incidences in a result yields a correct 'dual' result.

To represent lines in P^3 one has to give up the normalization of Plücker coordinates since a line L at infinity (intersection of parallel planes) does not possess a direction vector.

To cover all cases, one spans a line L by two points $\mathsf{x}\mathbb{R}$, $\mathsf{y}\mathbb{R}$, possibly at infinity. Then, the *homogeneous Plücker coordinates* are found as

$$\begin{aligned} \mathsf{L} = \mathsf{x} \wedge \mathsf{y} \quad &:= \quad (x_0\mathbf{y} - y_0\mathbf{x}; \mathbf{x} \times \mathbf{y}), \qquad (4) \\ = (\mathbf{l}, \bar{\mathbf{l}}) \quad &= \quad (l_1, l_2, l_3, l_4, l_5, l_6) \in \mathbb{R}^6, \end{aligned}$$

As the homogenized form of the previously discussed case of normalized Plücker coordinates, these coordinates again satisfy the Plücker relation $\mathbf{l}\cdot\bar{\mathbf{l}} = 0$. Lines L at infinity are characterized by $\mathbf{l} = \mathbf{o}$. Note that a line L at infinity is determined by parallel planes through it. A normal vector of these planes is $\bar{\mathbf{l}}$. If L is not at infinity, it can be spanned by a point in E^3 with coordinates $\mathsf{x} = (1, \mathbf{x})$ and a point at infinity $\mathsf{y} = (0, \mathbf{y})$. Inserting in (4) we see that $\mathbf{l} = \mathbf{y}$ is a (not necessarily normalized) direction vector of L and $\bar{\mathbf{l}} = \mathbf{x}\times\mathbf{l}$ is the moment vector of the force $\mathbf{l}$ on L about the origin.

The interpretation of a line L as connection of two points $\mathsf{x}\mathbb{R}$, $\mathsf{y}\mathbb{R}$ can be dualized to the interpretation of a line L as intersection of two planes $\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, \mathbf{u})$ and $\mathbb{R}\mathsf{v} = \mathbb{R}(v_0, \mathbf{v})$. Then the homogeneous Plücker coordinates of the intersection line are

$$\mathsf{L} = (\mathbf{l}, \bar{\mathbf{l}}) = (\mathbf{u}\times\mathbf{v}, u_0\mathbf{v} - v_0\mathbf{u}).$$

If $L = \mathbb{R}\mathsf{u}\cap\mathbb{R}\mathsf{v}$ is not at infinity, $\mathbf{u}\times\mathbf{v}$ is a direction vector of L.

The rule of duality says that exchanging the meaning of point and plane coordinates in a formula and exchanging vectors $\mathbf{l}$ and $\bar{\mathbf{l}}$ in the Plücker coordinates, gives a correct dual formula.

Example: One can prove that the intersection points $\mathsf{s}_i\mathbb{R}$ of a line L with the coordinate planes $x_i = 0$ are

$$\begin{array}{llll} \mathsf{s}_0 = & (0, l_1, l_2, l_3), & \mathsf{s}_1 = & (-l_1, 0, l_6, -l_5), \\ \mathsf{s}_2 = & (-l_2, -l_6, 0, l_4), & \mathsf{s}_3 = & (-l_3, l_5, -l_4, 0). \end{array}$$

To verify this, we compute the Plücker coordinates from any pair of points $\mathsf{s}_i, \mathsf{s}_j$ via (4) and, possibly making use of homogeneity and the Plücker relation, we find in all cases $(l_1, \ldots, l_6)$. Two of the four points s_i are certainly different. If a line L lies in a plane $\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, \mathbf{u})$, all points $\mathsf{s}_i\mathbb{R}$ lie in that plane. These four linear equations, where only two are necessary, can be collected to

$$\mathbf{l}\cdot\mathbf{u} = 0, -u_0\mathbf{l} + \mathbf{u}\times\bar{\mathbf{l}} = \mathbf{o}, \tag{5}$$

which is the *incidence condition* of a line L and a plane $\mathbb{R}\mathsf{u}$. Dual to that one gets the incidence condition of a line L and a point $\mathsf{x}\mathbb{R}$ by

$$\bar{\mathbf{l}}\cdot\mathbf{x} = 0, -x_0\bar{\mathbf{l}} + \mathbf{x}\times\mathbf{l} = \mathbf{o}. \tag{6}$$

Example: The intersection point $\mathsf{p}\mathbb{R} = (p_0, \mathbf{p})\mathbb{R}$ of a line L and a plane $\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, \mathbf{u})$ can be calculated as follows. Let $\mathsf{s}_i, \mathsf{s}_j$ be two distinct points and $\lambda\mathsf{s}_i + \mu\mathsf{s}_j$ be a parametrization of L in homogeneous parameters. The intersection point satisfies $\lambda(\mathsf{s}_i\cdot\mathsf{u}) + \mu(\mathsf{s}_j\cdot\mathsf{u}) = 0$, which leads to homogeneous parameters $(\lambda : \mu) = (-\mathsf{s}_j\cdot\mathsf{u} : \mathsf{s}_i\cdot\mathsf{u})$. Inserting this, some algebra leads to the formula of the intersection point

$$(p_0, \mathbf{p}) = (\mathbf{u}\cdot\mathbf{l}, -u_0\mathbf{l} + \mathbf{u}\times\bar{\mathbf{l}}). \tag{7}$$

Dual to this the connecting plane $\mathbb{R}(u_0, \mathbf{u})$ of a point $\mathsf{x}\mathbb{R}$ and a line L is given by

$$(u_0, \mathbf{u}) = (\mathbf{x}\cdot\bar{\mathbf{l}}, -x_0\bar{\mathbf{l}} + \mathbf{x}\times\mathbf{l}). \tag{8}$$

Let s_i, s_j and t_i, t_j be two distinct pairs of points in coordinate planes on lines $G = (\mathbf{g}, \bar{\mathbf{g}})\mathbb{R}$ and $H = (\mathbf{h}, \bar{\mathbf{h}})\mathbb{R}$, respectively. The two lines G and H intersect, iff the four mentioned points lie in a common plane. This can be expressed by

$$\det(\mathsf{s}_i, \mathsf{s}_j, \mathsf{t}_i, \mathsf{t}_j) = 0.$$

Inserting the Plücker coordinates G and H it follows that the *lines G and H intersect iff*

$$\begin{aligned} 0 &= g_1h_4 + g_2h_5 + g_3h_6 + g_4h_1 + g_5h_2 + g_6h_3 \\ &= \mathbf{g}\cdot\bar{\mathbf{h}} + \bar{\mathbf{g}}\cdot\mathbf{h}. \end{aligned} \tag{9}$$

Sometimes, we will briefly denote the bilinear form $\mathbf{g}\cdot\bar{\mathbf{h}} + \bar{\mathbf{g}}\cdot\mathbf{h}$ in this intersection condition by $\Omega(\mathsf{G}, \mathsf{H})$.

We want to derive a formula for the intersection point $\mathsf{s}\mathbb{R} = (s_0, \mathbf{s})\mathbb{R}$ of intersecting lines G and H with Plücker coordinates $(\mathbf{g}, \bar{\mathbf{g}})$ and $(\mathbf{h}, \bar{\mathbf{h}})$. Using the incidence condition (6), we obtain

$$\begin{aligned} \bar{\mathbf{g}}\cdot\mathbf{s} = 0, -s_0\bar{\mathbf{g}} + \mathbf{s}\times\mathbf{g} = \mathbf{o}, \\ \bar{\mathbf{h}}\cdot\mathbf{s} = 0, -s_0\bar{\mathbf{h}} + \mathbf{s}\times\mathbf{h} = \mathbf{o}. \end{aligned}$$

Assuming that $\bar{\mathbf{g}}, \bar{\mathbf{h}}$ are linearly independent, the two left hand side equations require $\mathbf{s} = \lambda\bar{\mathbf{g}}\times\bar{\mathbf{h}}$, $\lambda \neq 0$. Further, one calculates s_0 and finally we obtain

$$(s_0, \mathbf{s}) = (-\mathbf{g}\cdot\bar{\mathbf{h}}, \bar{\mathbf{g}}\times\bar{\mathbf{h}}) = (\bar{\mathbf{g}}\cdot\mathbf{h}, \bar{\mathbf{g}}\times\bar{\mathbf{h}}). \tag{10}$$

We get two formulas since $\Omega(\mathsf{G}, \mathsf{H}) = 0$ holds. But note that the linear independence of $\bar{\mathbf{g}}$ and $\bar{\mathbf{h}}$ is the crucial point in this formula. It does not work in general if one line, say G passes through the origin ($\bar{\mathbf{g}} = \mathbf{o}$). In this case a direct computation leads to

$$(s_0, \mathbf{s}) = ((\mathbf{g}\times\mathbf{h})\cdot(\mathbf{g}\times\mathbf{h}), \det(\mathbf{g}, \mathbf{h}, \bar{\mathbf{h}})\mathbf{g}).$$

Formula (10) does not work also for parallel lines ($\mathbf{g} = \lambda\mathbf{h}$), but in this case the intersection point is known as $\mathsf{s}\mathbb{R} = (0, \mathbf{g})\mathbb{R}$.

Figure 2: Subspaces on the Klein quadric

The Klein mapping

The homogeneous Plücker coordinates define a mapping from lines L in P^3 to ordered, homogeneous 6–tupels

$$\mathsf{L} = (\mathbf{l}, \bar{\mathbf{l}}) \in \mathbb{R}^6 \setminus \{\mathsf{O}\}.$$

These 6–tupels may be interpreted as points in 5-dimensional real projective space P^5. We write $\mathsf{L}\mathbb{R}$ to indicate homogeneity and also denote points in P^5 in this way. This means that we obtain a mapping from the set of lines in P^3 to points in P^5, which is called *Klein mapping.* Not all points in P^5 are Klein images of lines. By the Plücker relation (2), exactly those points in P^5 are images of lines in P^3 under the Klein mapping γ, which lie on the *Klein quadric* M_2^4

$$l_1 l_4 + l_2 l_5 + l_3 l_6 = 0. \qquad (11)$$

The quadratic form $\mathbf{l} \cdot \bar{\mathbf{l}} = l_1 l_4 + \ldots + l_3 l_6$ will also be written as $\Omega(\mathsf{L})$.

The Klein mapping γ has the following important properties. *The lines of a pencil are mapped to the points of a line.* This can be seen in the following way. Let a pencil of lines be spanned by two intersecting lines L_1, L_2 with intersection point $\mathsf{x}\mathbb{R}$. Their Plücker coordinates shall be L_1, L_2 and let $\mathsf{y}_1\mathbb{R}$, $\mathsf{y}_2\mathbb{R}$ be a point on L_1 and L_2, respectively. An arbitrary line L of this pencil can be spanned by the vertex $\mathsf{x}\mathbb{R}$ and a point $\mathsf{y}\mathbb{R}$ on the line through $\mathsf{y}_1\mathbb{R}$, $\mathsf{y}_2\mathbb{R}$. Such a point has a coordinate vector $\mathsf{y} = \lambda_1 \mathsf{y}_1 + \lambda_2 \mathsf{y}_2$. Hence, the Plücker coordinates of L are

$$\begin{aligned} \mathsf{L} &= \mathsf{x} \wedge (\lambda_1 \mathsf{y}_1 + \lambda_2 \mathsf{y}_2) \\ &= \lambda_1(\mathsf{x} \wedge \mathsf{y}_1) + \lambda_2(\mathsf{x} \wedge \mathsf{y}_2) = \lambda_1 \mathsf{L}_1 + \lambda_2 \mathsf{L}_2. \end{aligned}$$

This says that the Klein image $\mathsf{L}\mathbb{R}$ lies on a line spanned by $\mathsf{L}_1\mathbb{R}$ and $\mathsf{L}_2\mathbb{R}$ on M_2^4, see figure 2.

Further, *the lines of a bundle are mapped to points of a plane, entirely contained in* M_2^4. A bundle consists of all lines through a fixed point, say $\mathsf{x}\mathbb{R}$. Let L_1, L_2, L_3 be three lines in the bundle. Their Plücker coordinates shall be L_1, $\mathsf{L}_2, \mathsf{L}_3$ and let $\mathsf{y}_1\mathbb{R}$, $\mathsf{y}_2\mathbb{R}$, $\mathsf{y}_3\mathbb{R}$ a point on each line. An arbitrary line L of this bundle can be spanned by the vertex $\mathsf{x}\mathbb{R}$ and a point $\mathsf{y}\mathbb{R}$ in the plane spanned by $\mathsf{y}_1\mathbb{R}$, $\mathsf{y}_2\mathbb{R}$ and $\mathsf{y}_3\mathbb{R}$, i.e. $\mathsf{y} = \lambda_1 \mathsf{y}_1 + \lambda_2 \mathsf{y}_2 + \lambda_3 \mathsf{y}_3$. The Plücker coordinates of L are therefore

$$\begin{aligned} \mathsf{L} &= \mathsf{x} \wedge (\lambda_1 \mathsf{y}_1 + \lambda_2 \mathsf{y}_2 + \lambda_3 \mathsf{y}_3) \\ &= \lambda_1(\mathsf{x} \wedge \mathsf{y}_1) + \lambda_2(\mathsf{x} \wedge \mathsf{y}_2) + \lambda_3(\mathsf{x} \wedge \mathsf{y}_3) \\ &= \lambda_1 \mathsf{L}_1 + \lambda_2 \mathsf{L}_2 + \lambda_3 \mathsf{L}_3. \end{aligned}$$

This says that the Klein image points $\mathsf{L}\mathbb{R}$ to the lines in a bundle are forming a plane. Of course, this plane is contained in the Klein quadric.

Using similar arguments we see that the lines lying in a fixed plane are also mapped to points of a plane, entirely contained in M_2^4. Thus, the Klein quadric M_2^4 carries exactly *two 3–parameter families* of two–dimensional planes, which are the γ–images of line bundles and line fields (lines in a fixed plane), respectively.

Two planes on M_2^4 belong to the same family, iff they have a point in common. This point is the Klein image of the connecting line of two bundles or the Klein image of the intersection line of two fields.

Otherwise two planes on M_2^4 intersect in a line or are skew, depending on whether the vertex of the bundle is contained in the plane of the field of lines or not.

A regular quadric in P^5 cannot contain 3–dimensional subspaces and thus M_2^4 carries maximal dimensional subspaces.

Example: All lines in P^3 passing through the origin satisfy $\bar{\mathbf{l}} = \mathbf{o}$ (see (1)). This means that their γ–images are contained in the plane $l_4 = l_5 = l_6 = 0$, which entirely lies on M_2^4.

Example: Let us consider lines in P^3 lying in the plane at infinity $x_0 = 0$. With formula (4) it follows that they satisfy $\mathbf{l} = \mathbf{o}$. This means that their γ–images lie in the plane $l_1 = l_2 = l_3 = 0$, entirely contained in M_2^4.

Special sets of lines

We will give a short overview of special sets of lines in Euclidean or projective 3–space. Especially problems in *constraint solving* [9] lead us to interesting examples. Detailed studies follow later in connection with space kinematics.

One-parameter families of lines

We have already got to know pencils of lines which are the simplest one-parameter sets of lines. Their Klein images are lines on M_2^4. One-parameter families of lines form ruled surfaces. Their Klein images are curves on M_2^4 and will be discussed in detail in a separate contribution [23]. Here we will give just some basic examples to get a better understanding of the Klein mapping.

Let a conic c in P^5 be the intersection of a 2–plane ϵ with M_2^4. We note that ϵ is not contained in M_2^4. It can be shown that the points of c are Klein images of lines on a ruled quadric Φ in P^3, precisely, one family of generator lines of Φ.

Example: Let a ruled quadric Φ be given by the equation $-x_0^2 + x_1^2 + x_2^2 - x_3^2 = 0$. The Plücker coordinates of one family of generator lines are

$$\mathsf{G}(t) = (-\sin t, \cos t, 1, \sin t, -\cos t, 1),\ t \in [0, 2\pi).$$

Any two lines $G(t_1)$, $G(t_2)$, $t_1 \neq t_2$ are skew. The plane ϵ containing the Klein images of these lines is

$$\epsilon : x_1 = -x_4, x_2 = -x_5, x_3 = x_6.$$

By the way, Φ carries a second family of generator lines

$$\mathsf{H}(s) = (-\sin s, \cos s, -1, -\sin s, \cos s, 1),\ \ s \in [0, 2\pi),$$

and, by (9), each line $G(t)$ intersects each line $H(s)$. The Klein images $\mathsf{H}(s)\mathbb{R}$ lie in a plane ϕ which is skew to ϵ.

If the conic c lies in a plane ϵ, entirely contained in M_2^4, its points are Klein images of generator lines of a quadratic cone or tangent lines of a conic in P^3, depending on whether ϵ is Klein image of a bundle or a field of lines.

Example: Let a quadratic cone be given by the equation $x_1^2 + x_2^2 - x_3^2 = 0$. The Plücker coordinates of the generator lines are

$$\mathsf{L}(t) = (\cos t, \sin t, 1, 0, 0, 0),\ t \in [0, 2\pi).$$

This defines the homogeneous parameter representation of a conic in M_2^4 which lies in the plane $x_4 = x_5 = x_6 = 0$. As another example,

$$\mathsf{L}(t) = (0, 0, 0, \cos t, \sin t, 1),\ t \in [0, 2\pi)$$

parameterizes the tangent lines of the conic $x_0 = 0$, $x_1^2 + x_2^2 - x_3^2 = 0$, lying in the plane at infinity. Note that these are the lines at infinity of all planes which form the constant angle $\pi/4$ with the z–axis. Again we see that the Klein image of this line set is a conic.

Two-parameter families of lines

The simplest two-parameter families are *bundles* and *fields* of lines and their Klein images are planes, entirely contained in M_2^4. Further, the normals of a surface in 3–space form a two-parameter family of lines, called *normal congruence*. General two-parameter families are called *line congruences* and their Klein images are a two-dimensional manifolds on M_2^4. Later, when we study more projective line geometry, we will have more examples. For instance, the intersections of M_2^4 with 3–spaces, that are two-dimensional quadrics in M_2^4, are Klein images of important families of lines (linear congruences).

Three-parameter families of lines

Three-parameter families of lines are much harder to imagine and we give two examples which come from *constraint solving*.

Example: We are searching for all lines L, which have a constant distance d from a given point $\mathbf{p}$. Without loss of generality let $d = 1$ and $\mathbf{p}$ be the origin of the coordinate system. The family of lines L we are considering are just all tangent lines of the unit sphere S^2. Using normalized Plücker coordinates $(\mathbf{l}, \bar{\mathbf{l}})$ for a moment, we first note that the length $||\bar{\mathbf{l}}||$ gives the distance of the line to the origin. Hence, the lines under discussion satisfy $\bar{\mathbf{l}}^2 = 1$ in normalized Plücker coordinates, which, by the normalization, is equivalent to $\bar{\mathbf{l}}^2 = \mathbf{l}^2$. Obviously, this homogeneous quadratic equation is already the desired relation in homogeneous Plücker coordinates,

$$V_2^4 : \mathbf{l}\cdot\mathbf{l} - \bar{\mathbf{l}}\cdot\bar{\mathbf{l}} = 0.$$

It defines the quadratic hypersurface V_2^4 in P^5. The upper index denotes the dimension, the lower index the degree. This means that the Klein images of tangent lines of the unit sphere lie in the intersection $M_2^4 \cap V_2^4$.

In general, such families of lines satisfying a quadratic equation in addition to the Plücker relation are called *quadratic complexes*. The tangent line complexes of quadrics are examples of quadratic complexes, but there are other types as well.

Example: We are interested in all lines L which intersect a given circle, say the unit circle in the xy-plane. It is easy to see that the Plücker coordinates of lines L satisfy $-l_3^2 + l_4^2 + l_5^2 = 0$. Again, we have a quadratic complex.

A further example of the same type is the following. We are interested in all lines L which enclose an angle of $\pi/4$ with the xy-plane. In Plücker coordinates $(\mathbf{l}, \bar{\mathbf{l}})$, the first vector $\mathbf{l}$ is a direction vector, and therefore these lines satisfy $l_1^2 + l_2^2 - l_3^2 = 0$ and again form a quadratic complex.

Above examples have led to quadratic complexes. What about the lines which satisfy a linear equation in Plücker coordinates? Using formula (9) we see that all lines H intersecting a fixed line G are given by a linear equation. But (9) is just a special case of a linear equation in Plücker coordinates, since both G and H satisfy the Plücker relation.

We use the notation

$$\mathsf{C} = (\mathbf{c}, \bar{\mathbf{c}}) = (c_1, c_2, c_3, c_4, c_5, c_6).$$

where $\mathbf{c}$, $\bar{\mathbf{c}}$ are not necessarily Plücker coordinates of a line. A set of lines $L = (\mathbf{l}, \bar{\mathbf{l}})$ satisfying a linear equation in Plücker coordinates,

$$c_4 l_1 + c_5 l_2 + c_6 l_3 + c_1 l_4 + c_2 l_5 + c_3 l_6 = 0 \qquad (12)$$

is called a *linear line complex* or linear complex. The strange numbering of the coefficients c_i will be explained later, (see formulae (15)). The expression on the left is $\bar{\mathbf{c}}\cdot\mathbf{l} + \mathbf{c}\cdot\bar{\mathbf{l}}$, which will also be written as $\Omega(\mathsf{C}, \mathsf{L})$.

There are two different types of linear complexes. Firstly, if $(\mathbf{c}, \bar{\mathbf{c}})$ are Plücker coordinates of a line C, the linear complex consists of all lines intersecting C, see (9). Such a linear complex is called *singular*.

Secondly, $\mathbf{c}\cdot\bar{\mathbf{c}} \neq 0$ characterizes *regular linear complexes*. So far we have no idea of such sets of lines and cannot explain their properties. Later they will be discussed in connection with kinematics and with so-called null polarities.

We want to motivate the study of line geometry by giving more examples on complexes. A general *algebraic line complex* is defined by a homogeneous algebraic equation $G(\mathbf{l}, \bar{\mathbf{l}}) = 0$ in Plücker coordinates, clearly different from (11). For our examples only those complexes are used which we already have understood.

Intersection of complexes

The intersection of two algebraic complexes in general position is an algebraic line congruence, the intersection of three algebraic complexes in general position is an algebraic ruled surface. This seems to be rather theoretical but the following examples shall demonstrate the practical use of this interpretation.

Let A_i, $i = 1, 2, 3$ be three given skew lines in 3–space. All lines L intersecting A_i shall be calculated. By the intersection condition (9) one obtains the Plücker coordinates of L as solutions of

$$\begin{aligned} \Omega(\mathsf{A}_i, \mathsf{L}) &= \bar{\mathbf{a}}_i\cdot\mathbf{l} + \mathbf{a}_i\cdot\bar{\mathbf{l}} = 0, \text{ for } i = 1, 2, 3, \\ \Omega(\mathsf{L}) &= \mathbf{l}\cdot\bar{\mathbf{l}} = 0. \end{aligned}$$

Thus, the lines L lie in the intersection of three singular linear complexes. The last equation just guarantees that $(\mathbf{l}, \bar{\mathbf{l}})$ are Plücker coordinates of lines L.

Interpreting these equations in P^5 we see that the Klein images $\mathsf{L}\mathbb{R}$ form an intersection of M_2^4 with a two-dimensional plane which is the intersection of the three hyperplanes $\Omega(A_i, L) = 0$, $i = 1, 2, 3$. Thus, the Klein images $\mathsf{L}\mathbb{R}$ of the considered lines lie on a conic. Using the result of a previous example it follows that the lines L form one family of generator lines of a ruled quadric Φ. The quadric Φ contains a second family of lines G, which contains the lines A_i and all lines G intersect all lines L.

Example: Let the lines A_i be given by their Plücker coordinates

$$\begin{aligned}
\mathsf{A}_1 &= (1,0,0,0,0,0),\\
\mathsf{A}_2 &= (0,1,0,-1,0,0),\\
\mathsf{A}_3 &= (0,0,1,1,-1,0).
\end{aligned}$$

All points in P^5 satisfying $\Omega(\mathsf{A}_i,\mathsf{X})=0$ form 2–plane ϵ, parametrized in homogeneous parameters by

$$\mathsf{Y}=(\mathbf{y},\bar{\mathbf{y}})=(\lambda,\mu,\nu,0,\lambda,-\lambda+\mu).$$

Only those points in ϵ are Klein images of lines $L\in P^3$ which lie on M_2^4. Thus we solve $\mathbf{y}\cdot\bar{\mathbf{y}}=0$ and this leads to the expected quadratic parameterization

$$\mathsf{L}(t)=(t(1+t),t,1+t,0,t(1+t),-t^2).$$

Further, the family $G(s)$ containing A_i is given by

$$\mathsf{G}(s)=(1-s,s(1-s),s,s^2,-s,0).$$

Clearly we have $A_1=G(0)$, $A_2=G(\infty)$ and $A_3=G(1)$ and $L(t)$, $G(s)$ are both families of generator lines on the ruled quadric determined by A_i.

Another constraint solving problem is the following. We are looking for all lines L which intersect a given line G orthogonally and have a certain distance d from a fixed point $\mathbf{p}$. From a previous example we know already that lines having a constant distance from a point form a quadratic complex. We choose again $d=1$ and $\mathbf{p}$ as origin of the coordinate system. Together with the further constraints we obtain the system

$$\begin{aligned}
V_2^4: \mathbf{l}\cdot\mathbf{l}-\bar{\mathbf{l}}\cdot\bar{\mathbf{l}} &= 0,\\
\mathbf{g}\cdot\bar{\mathbf{l}}+\bar{\mathbf{g}}\cdot\mathbf{l} &= 0,\\
\mathbf{g}\cdot\mathbf{l} &= 0,\\
M_2^4: \mathbf{l}\cdot\bar{\mathbf{l}} &= 0.
\end{aligned}$$

The intersection of these four hypersurfaces (M_2^4 and V_2^4 are quadratic, the other two are linear) in P^5 is, by Bézout's theorem, in general a curve of order 4. We will see later that this curve is the Klein image of a ruled surface of order 4, which is the solution of our problem.

So far we have studied the correspondence between lines L in 3–space and their Klein images $\mathsf{L}\mathbb{R}$ in P^5. We have got a point model (M_2^4) for the set of lines, which is a helpful tool for solving certain problems. The coordinates of points $\mathsf{C}\mathbb{R}\in P^5$ not on the Klein quadric, i.e. $\mathbf{c}\cdot\bar{\mathbf{c}}\neq 0$, appear naturally in connection with regular linear complexes (12). But to get a better understanding of linear complexes and those points $\mathsf{C}\mathbb{R}$ we need more projective geometry. This will turn out as very useful in a study of rational ruled surfaces [23]. However, before extending our tutorial on the projective theory, we would like to draw a picture of the nice relation between linear complexes and kinematics and use it for some applications in surface reconstruction and robot kinematics.

Line geometry and kinematics

The relation between line geometry and kinematics in E^3 will help us to understand linear complexes from a practical point of view.

A continuous motion in E^3, composed of a continuous rotation around a line A (with constant angular velocity) and a continuous translation parallel to A (with constant velocity) is called a *helical motion* or *screw motion* with *axis* A and *pitch* p, see figure 3. If A is chosen to be the z–axis of a Cartesian coordinate system, a helical motion is given by

$$\mathbf{x}(t)=\begin{pmatrix}0\\0\\pt\end{pmatrix}+\begin{pmatrix}\cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\mathbf{x}(0),$$

but in the following this special choice of A is not used. For $p=0$, the motion reduces to a pure *rotation* around A. If $p\neq 0$, all points of the axis

Figure 3: Helical motion with axis A, pitch p

A travel along it, whereas the paths (trajectories) of the remaining points of the moving system are *helices*. As limit case $p = \infty$, we include a continuous *translation*, where the rotational part vanishes and all points travel with constant velocity along parallel lines.

The velocity vector field for these motions is time independent and linear, see Bottema and Roth [2]. A point $\mathbf{x}$ has velocity vector

$$\mathbf{v}(\mathbf{x}) = \bar{\mathbf{c}} + \mathbf{c} \times \mathbf{x}, \tag{13}$$

where the constant vectors $\mathbf{c}, \bar{\mathbf{c}} \in \mathbb{R}^3$ are related to axis $A = (\mathbf{a}, \bar{\mathbf{a}})$ and pitch p of a helical or rotational motion via

$$p = \frac{\mathbf{c} \cdot \bar{\mathbf{c}}}{\mathbf{c}^2}, \quad (\mathbf{a}, \bar{\mathbf{a}}) = (\mathbf{c}, \bar{\mathbf{c}} - p\mathbf{c}). \tag{14}$$

Here, $\mathbf{c}$ is not necessarily normalized. We note that $\mathbf{c} = \mathbf{o}$ belongs to translations parallel to $\bar{\mathbf{c}}$.

Lines through points $\mathbf{x}$ normal to $\mathbf{v}(\mathbf{x})$ are normal to the trajectory at $\mathbf{x}$ and are called *path normals*. Their Plücker coordinates satisfy a linear equation

$$\begin{aligned} 0 &= \mathbf{v} \cdot \mathbf{l} = \bar{\mathbf{c}} \cdot \mathbf{l} + (\mathbf{c} \times \mathbf{x}) \cdot \mathbf{l} = \\ &\quad \bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot (\mathbf{x} \times \mathbf{l}) = \bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot \bar{\mathbf{l}}. \end{aligned} \tag{15}$$

We have called such three parameter sets linear complexes, defined by the coefficient vectors $\mathbf{c}$ and $\bar{\mathbf{c}}$. Hence, the *path normals of a helical motion, rotation or translation form a linear complex* $\mathcal{C} = (\mathbf{c}, \bar{\mathbf{c}})$. But note that in general $\mathbf{c} \cdot \bar{\mathbf{c}} \neq 0$, such that $(\mathbf{c}, \bar{\mathbf{c}})$ are not Plücker coordinates of a line in E^3.

Conversely, any linear complex in E^3, defined by vectors $\mathbf{c}, \bar{\mathbf{c}}$, defines a velocity vector field (13), and thus is formed by the path normals of a helical (rotational, translational) motion. We have called a linear complex singular, iff the coefficient vectors $\mathbf{c}, \bar{\mathbf{c}}$ satisfy the Plücker relation $\mathbf{c} \cdot \bar{\mathbf{c}} = 0$, and $(\mathbf{c}, \bar{\mathbf{c}})$ are (not necessarily normalized) Plücker coordinates of a line.

Using the kinematical interpretation, singular complexes belong to $p = 0$ and $p = \infty$. In the case $p = 0$ the linear complex consists of all lines intersecting the rotational axis $A = (\mathbf{c}, \bar{\mathbf{c}})$, whereas in the case $p = \infty$ it consists of all lines which are perpendicular to the vector $\bar{\mathbf{c}}$, which means that these lines intersect a line at infinity.

Figure 4: Path normals of a helical/rotational motion with axis A

Since a linear complex determines a helical, rotational or translational motion, we also speak of pitch p and axis A of the linear complex $\mathcal{C}$, which are found via (14) in the non–translational case.

At any non–stationary instant of a one–parameter motion in E^3, the velocity vector field is of the form (13) and thus the instantaneous path normals form a linear complex.

Approximation in Line Space

In practice, errors in data are often unavoidable. In certain applications which will be discussed here the question arises how to construct a linear complex $\mathcal{C}$, which — in a sense to be specified — best approximates a given set of lines L_i, $i = 1, \ldots, k$. In other words, we are interested in the construction of a *linear complex of regression* to a given set of data lines.

An important input to the solution of the problem is an appropriate measure of the deviation of a given line L from a given linear complex $\mathcal{C}$ in E^3. Let us represent L with *normalized* Plücker coordinates $(\mathbf{l}, \bar{\mathbf{l}})$, $||\mathbf{l}|| = 1$. The linear complex $\mathcal{C}$ shall be defined by the equation $\bar{\mathbf{c}} \cdot \mathbf{x} + \mathbf{c} \cdot \bar{\mathbf{x}} = 0$. For the moment, the linear complex $\mathcal{C}$ shall be different from the path normal complex of a translation and thus satisfy $\mathbf{c} \neq \mathbf{o}$.

According to F. Klein [17] we use the *moment of L with respect to $\mathcal{C}$*,

$$m(L, \mathcal{C}) := \frac{|\bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot \bar{\mathbf{l}}|}{||\mathbf{c}||} \tag{16}$$

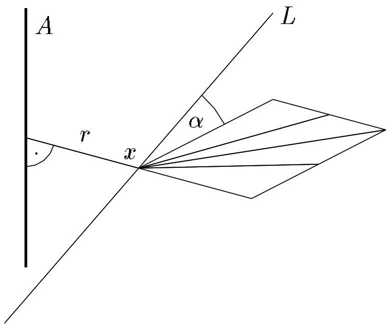

Figure 5: Moment of a line w.r.t. a linear complex

to measure the deviation of L from $\mathcal{C}$. Usually one defines the moment between an oriented line and an oriented linear complex (linear complex with oriented axis) and thus omits the absolute value in the definition. We do not need orientations and thus define the moment as a nonnegative real number.

The moment has the following geometric interpretation, see figure 5. Pick an arbitrary point $\mathbf{x}$ on L. Let r be distance of $\mathbf{x}$ from the axis A of the complex $\mathcal{C}$ with pitch p. Let $\alpha \in [0, \pi/2]$ be the smallest angle between L and a line of $\mathcal{C}$ through $\mathbf{x}$, or, in other words, the angle between L and the plane (path normal plane of $\mathbf{x}$), in which the lines of $\mathcal{C}$ through $\mathbf{x}$ are lying, see figure 5. Then, the moment m can also be expressed as

$$m(L, \mathcal{C}) = \sqrt{r^2 + p^2} \sin \alpha. \tag{17}$$

Lines with vanishing moment, $m(L, \mathcal{C}) = 0$, are exactly the lines of $\mathcal{C}$. The formula above indicates that the moment is a useful deviation measure of a line from a linear complex (pretty much like the distance of a point to a plane in 3–space).

Linear complex of regression

Let us now compute a linear complex $\mathcal{C}$ which is as close as possible to the given lines L_i, $i = 1, \ldots, k$ with normalized Plücker coordinates $(\mathbf{l}_i, \bar{\mathbf{l}}_i)$. For that we compute $\mathcal{C}$ as minimizer of

$$\sum_{i=1}^{k} m(L_i, \mathcal{X})^2 \tag{18}$$

among all linear complexes $\mathcal{X}$, represented by $\mathsf{X} = (\mathbf{x}, \bar{\mathbf{x}}) \in \mathbb{R}^6$. With (16) this is equivalent to minimizing the positive semidefinite quadratic form

$$F(\mathsf{X}) := \sum_{i=1}^{k} (\bar{\mathbf{x}} \cdot \mathbf{l}_i + \mathbf{x} \cdot \bar{\mathbf{l}}_i)^2 =: \mathsf{X}^T \cdot M \cdot \mathsf{X} \tag{19}$$

under the normalization condition

$$1 = ||\mathbf{x}||^2 =: \mathsf{X}^T \cdot D \cdot \mathsf{X}, \tag{20}$$

where $D = \mathrm{diag}(1, 1, 1, 0, 0, 0)$. This is a familiar general eigenvalue problem. Using a Lagrangian multiplier λ, we have to solve the system

$$(M - \lambda D) \cdot \mathsf{X} = 0, \ \mathsf{X}^T \cdot D \cdot \mathsf{X} = 1. \tag{21}$$

The first equation has a non trivial solution X, if λ is a root of the equation

$$\det(M - \lambda D) = 0. \tag{22}$$

Because three diagonal elements in D are zero, this is just a cubic equation in λ. For any root λ and corresponding general eigenvector $\mathsf{X} = (\mathbf{x}, \bar{\mathbf{x}})$, which is a solution of $(M - \lambda D) \cdot \mathsf{X} = 0$ and with $||\mathbf{x}|| = 1$, we have

$$F(\mathsf{X}) = \mathsf{X}^T \cdot M \cdot \mathsf{X} = \lambda \mathsf{X}^T \cdot D \cdot \mathsf{X} = \lambda.$$

This says that all roots λ are nonnegative and the *solution* C *is a general eigenvector to the smallest general eigenvalue* $\lambda \geq 0$.

The standard deviation of the given lines L_i from the approximating linear complex $\mathcal{C}$ is

$$\sigma = \sqrt{\frac{\lambda}{k-5}}. \tag{23}$$

In case of a sufficiently small deviation σ, the given lines L_i can be well approximated by lines of the linear complex $\mathcal{C}$. We use (14) to compute axis A and pitch p of the complex $\mathcal{C}$. Note that the moment m, the standard deviation σ as well as the pitch p are distances in Euclidean geometry and thus their magnitude has to be seen in comparison to the error tolerances and the size of the objects under discussion.

Linear complexes with pitch $p = \infty$ have so far been excluded. They are characterized by $\mathbf{c} = \mathbf{o}$ and the complex consists of all lines in E^3 which are orthogonal to the vector $\bar{\mathbf{c}}$. The deviation of a line L from such a complex can simply be taken as cosine of the angle between L and $\bar{\mathbf{c}}$. Thus, we now minimize

$$\sum_{i=1}^{k}(\bar{\mathbf{x}} \cdot \mathbf{l}_i)^2 \tag{24}$$

over all unit vectors $\bar{\mathbf{x}} \in \mathbb{R}^3$. This is an eigenvalue problem in $\mathbb{R}^3$. Note that one might not know in advance whether an approximation of the given data lines L_i with this special type of a singular linear complex would be possible. If it is possible, the general algorithm as outlined above will cause numerical problems since all coefficients in (22) will be close to zero. There are two simple ways to overcome this problem. One can either check for a complex with $p = \infty$ first or run at first the general algorithm with another normalization condition, namely $\mathsf{X}^2 = 1$. The latter case is equivalent to setting D as unit matrix. Note, however, that this leads to a characteristic equation (22) of degree 6.

Families of solution complexes: It might be that in solving (22) two small eigenvalues λ_1, λ_2 occur. Then there exist two nearly equally good solution complexes $\mathcal{C}_1, \mathcal{C}_2$ to the given set of lines L_i. Thus, the lines L_i can be well approximated by lines of the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$. This is a special two parameter family of lines, which shall be discussed later in the section on 'The Klein image of a linear complex'.

Analogously, three small eigenvalues $\lambda_1, \lambda_2, \lambda_3$ define three linear complexes. The intersection of independent complexes $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ is in general a one parameter family of generator lines of a ruled quadric, see also the section 'The Klein image of a linear complex'.

A detailed treatment of special cases and the approximation of line segments by lines in a linear complex is done by Pottmann *et al.* [25].

Rotational / helical surface reconstruction

Let us look at a problem that arises in the context of *reverse engineering* of geometric models[33]. There, a real part has to be transformed into a computer model, including a CAD description of its geometry. The surface of a part may consist of different surface types. There might be simple surfaces like planes, spheres, cones and cylinders of revolution and tori. It might also contain more general surfaces of revolution, general cylinders and helical surfaces and it might exhibit general freeform surfaces. Both for the CAD representation and for the manufacturing of the part, it is essential to recognize the simpler surface types and fit the given data, usually clouds of points with measurement errors, by surfaces of the determined type.

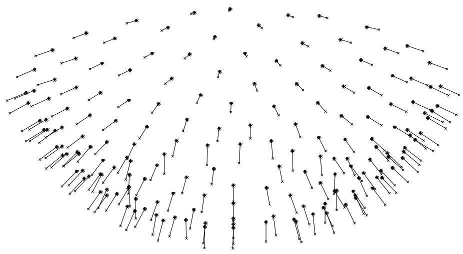

Figure 6: Perturbated data points of a rotational surface with estimated normal vectors

Recall that a general *cylinder* is traced out by a curve which is moved under a continuous (straight) translation. A *surface of revolution* is generated by a curve that is undergoing a continuous rotation. Since the simplest surface types are special cylinders or surfaces of revolution, it seems to be important to recognize whether the given data can be fitted well with a cylinder or a surface of revolution and to develop an algorithm for computing a fit with the surface of the determined type. The method of Pottmann and Randrup[26] then suggests to look more generally at helical surfaces, which are generated by curves under screw motions. The approach is based on the following well–known result. *The normals of a C^1 surface lie in a linear complex if and only if the surface is (part of) a cylinder, a surface of revolution or a helical surface.*

Let us now return to our problem, how to recognize helical and rotational surfaces. Let a set of scattered (noisy) data points of a helical surface Φ or one of its special or limit forms (surface of revolution, cylinder) be given.

1. Firstly, we have to estimate the surface normals N_i at the given data points $\mathbf{d}_i$, $i = 1, \ldots, k$,

see figure 6. For methods that can be employed to estimate surface normals, we refer to the CAGD literature[11]. Here one has to take care that the density of the points $\mathbf{d}_i$ at which we compute the normals N_i is approximately the same everywhere. For reverse engineering applications we will perform a data reduction algorithm[20] and compute the normals just on a subset of the original data points. We assume that $\mathbf{d}_i$ is already an appropriate set of data points.

2. Secondly, because of the fundamental result mentioned above we now have to calculate an approximating linear complex $\mathcal{C}$ to the estimated normals N_i. In case of a small deviation (23) of the normals N_i from $\mathcal{C}$, there will be a good helical surface approximant to the given data points $\mathbf{d}_i$. To determine the helical surface we use (14) to compute axis and pitch p of the generating motion. This gives important information for *type recognition*, since for instance the normals of a surface of revolution lie in a singular complex with $p = 0$. We will not pursue this here but refer the reader to the articles [25, 26, 28].

   In connection with type recognition it has to be mentioned, that cylinder surfaces, which would cause numerical problems in the algorithm, can be detected earlier. The unit normal vectors of a cylinder form a great circle (Gaussian image) on the unit sphere. So one should first check the Gaussian image of the surface before running the approximation algorithm.

3. So far, we have discussed the construction of the generating motion. Thirdly, we want to reconstruct the surface. What is missing is just a curve c, which generates the surface under the just defined motion. To determine c, we project (move) the given data points $\mathbf{d}_i$, $i = 1, \ldots, k$ with help of their trajectories into an appropriate plane π, see figure 7. For a surface of revolution, π passes through the axis, for a cylinder it is orthogonal to its generators and for a helical surface one can choose π as path normal plane of an appropriate data point [26].

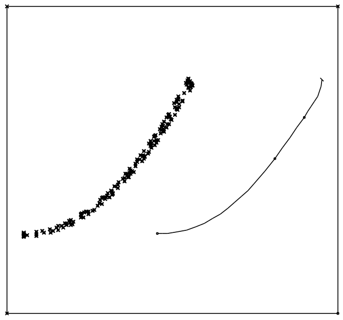

Figure 7: The auxiliary plane π with the projected points and approximation curve c (shifted)

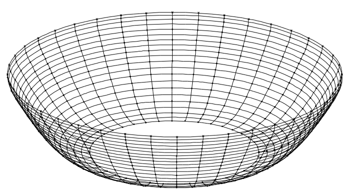

Figure 8: Approximating surface of the data points

   The resulting points $\mathbf{p}_i$ in π form a set, which should lie close to the curve c to be constructed. For computing an approximating curve c of the points $\mathbf{p}_i$, one can use the methods devised by Lee[18] or Randrup[28].

4. Finally, moving c with help of the determined motion yields the approximating surface (figure 8), which may have to be trimmed according to the region of given data.

On the stability of a parallel manipulator's position

Let us consider a *six–legged parallel manipulator*. Here, the moving system Σ and the fixed base system Σ^0 are connected by 6 legs, which are prismatic joints (hydraulic cylinders) attached to both systems via spherical joints. Geometrically, we may think of the legs L_i, $i = 1, \ldots, 6$ as lines. Special cases of such manipulators appear for example in flight simulators. Research on this kind of robots

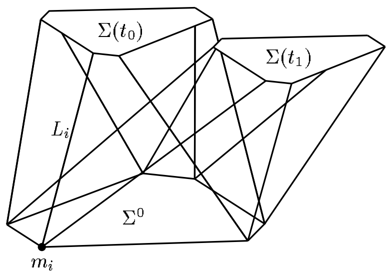

Figure 9: Parallel manipulator

has been quite active in the past decade[16]. We will show its relation to line geometry and in particular point to a *stability concept based on approximation in line space.*

Keeping the leg lengths fixed at a position $\Sigma(t)$ of the moving system, we should obtain a rigid system. However, it may be infinitesimally movable and even admit a finite motion. Such positions are referred to as *singular.* Since the legs L_i of fixed length r_i are fixed at points $\mathbf{m}_i \in \Sigma^0$, they can only move on spheres Φ_i with centers $\mathbf{m}_i$ and radii r_i. Hence, the legs appear as path normals and we immediately arrive at the following well–known result[16, 21]. *A position of a parallel robot with $k \geq 6$ legs is singular if and only if the positions of the legs lie in a linear complex.* In practice, such positions are avoided. Moreover, if a robot has two different, but in space nearby configurations for the same leg lengths, the robot may snap into the neighboring position, which is clearly undesirable. Such positions are close to a singular position [16]. Therefore snapping into a neighboring position can be avoided by avoiding *positions where the legs can be fitted well by a linear complex.* Note that we have given the computational tools for checking this. Moreover, various sources for errors occur in practice such that the test for a singular position itself should be formulated as a regression problem.

In case that the leg positions $L_i(t)$ lie in the intersection of two independent linear complexes, we even have an instantaneous movability of a two–parameter motion. Finally, if the legs lie in the intersection of three independent linear complexes, the indeterminacy of the points of the moving system is in three independent directions in general (in two directions for the spherical links)[21]. Again, our results are suitable to detect such positions [25].

An analogous application occurs in the context of *serial robots.* For example, if the robot has six revolute joints, a necessary (but not sufficient) condition for a singular position (defined by vanishing Jacobian of the mapping from the 6–dimensional configuration space to the motion group $SE(3)$) is that the positions of joint axes lie in a linear complex [15, 16].

More projective geometry

We have seen that the path normals of a helical (rotational, translational) motion lie in a regular (singular) linear complex. We will now study more projective geometry to get another interpretation for linear complexes in connection with so-called null polarities. Especially studying their Klein images will be important for dealing with rational ruled surfaces [23].

Let P^3 be projective 3–space, as above. We use homogeneous coordinates $\mathsf{x} = (x_0, \mathbf{x})$ to represent points $\mathsf{x}\mathbb{R}$. In the same way, $\mathsf{u} = (u_0, \mathbf{u})$ are homogeneous plane coordinates of a plane $\mathbb{R}\mathsf{u}$ with respect to a fixed coordinate system.

A regular linear map $f : P^3 \mapsto P^3$ from points $\mathsf{x}\mathbb{R}$ to points $\mathsf{y}'\mathbb{R}$ can be written with help of a regular 4×4–matrix A as

$$f : \mathsf{y}' = A \cdot \mathsf{x}. \tag{25}$$

Such a map is called a *projective map* or a *collineation.* These maps play a fundamental role in projective geometry, since projective geometry studies properties which are invariant under projective maps. Note that, by linearity, a collineation maps lines and planes (as point sets) to lines and planes, respectively.

In connection with our study of line geometry it is quite useful to know that a projective map induces a bijective transformation in the set of lines. This transformation is seen in the Klein image as a mapping of the Klein quadric M_2^4 onto itself; it can be shown that this mapping is the restriction of a collineation in P^5 onto the Klein quadric.

In view of the importance of linear mappings and the duality between points and planes it is natural to study also linear maps between the sets of points and planes. For that we use equation (25) and interpret y' as coordinate vector of a plane $\mathbb{R}\mathsf{y}'$. We get a linear map from points to planes, which is called a projective *correlation*. Linearity implies that points on a line are mapped to planes through a line. By the way, this induces a bijective map in the set of lines which is also seen in the Klein image as a collineation. Moreover, a correlation maps points in a plane to planes through a point; hence, in the explained sense, it maps planes to points. Summarizing, one can say that a correlation maps a projective figure to a *dual* counterpart of it. In this sense, it realizes the principle of duality we have come across earlier.

Example: A point $\mathsf{x}\mathbb{R}$ shall run on a conic c. Without loss of generality, let $\mathsf{x}(t) = (0, \cos t, \sin t, 1)$ be a parametrization of

$$c : x_0 = 0, x_1^2 + x_2^2 - x_3^2 = 0.$$

The correlation $\delta : \mathsf{u}' = I \cdot \mathsf{x}$ with I as identity matrix maps points $\mathsf{x}\mathbb{R}$ to planes $\mathbb{R}\mathsf{u}' = \mathbb{R}(0, \cos t, \sin t, 1)$, which envelop the cone $\delta(c) : x_1^2 + x_2^2 - x_3^2 = 0$. The origin, which is the vertex of $\delta(c)$ corresponds to the plane $x_0 = 0$ containing c. Moreover, the tangent lines of c are mapped to the generator lines of the cone $\delta(c)$.

Remark: Reading the foregoing example carefully, one will say that $x_1^2 + x_2^2 - x_3^2 = 0$ is not an equation in plane coordinates. The correct equation for $\delta(c)$ in plane coordinates is of course

$$\delta(c) : u_0 = 0, u_1^2 + u_2^2 - u_3^2 = 0,$$

which represents a one-parameter family of tangent planes. Nevertheless, this plane representation is converted to a point representation of $\delta(c)$ by computing the envelope of the plane set, which is $x_1^2 + x_2^2 - x_3^2 = 0$.

By the way, any developable surface, different from a plane, is representable as a one-parameter family of tangent planes and clearly as a two-parameter family of points.

Linear complexes and null polarities

Regular (singular) linear complexes occurred as path normal complexes of helical (rotational, translational) motions in Euclidean 3–space. This interpretation serves also to derive a definition of linear complexes in projective space.

Let a helical motion be defined by its velocity vectors (13). For a fixed point $\mathsf{z}\mathbb{R} = (1, \mathbf{z}/z_0)\mathbb{R}$ not at infinity the path normals form a pencil of lines in the *path normal plane* $\nu(\mathsf{z})$. Using formula (13), this plane is given by the equation

$$\begin{aligned} \nu(\mathsf{z}) : -(\bar{\mathbf{c}} \cdot \frac{\mathbf{z}}{z_0}) + (\bar{\mathbf{c}} + \mathbf{c} \times \frac{\mathbf{z}}{z_0}) \cdot \frac{\mathbf{x}}{x_0} &= 0 \\ \Longleftrightarrow (\mathbf{z} \cdot \bar{\mathbf{c}})x_0 + (-z_0\bar{\mathbf{c}} + \mathbf{z} \times \mathbf{c}) \cdot \mathbf{x} &= 0. \end{aligned}$$

We can use this equation to define a mapping $\varkappa$ from points $\mathsf{z}\mathbb{R}$ to planes $\nu(\mathsf{z})$. Obviously, it depends linearly on z and serves also to map points $(0, \mathbf{z})\mathbb{R}$ at infinity. Thus, this mapping is a correlation. If u' are homogeneous plane coordinates of the image plane $\nu(\mathsf{z})$, the correlation $\varkappa$, which is called *null polarity*, is given by

$$\mathsf{z} = (z_0, \mathbf{z}) \mapsto \mathsf{u}' = (\mathbf{z} \cdot \bar{\mathbf{c}}, -z_0\bar{\mathbf{c}} + \mathbf{z} \times \mathbf{c}). \qquad (26)$$

The image plane $\mathbb{R}\mathsf{u}'$ is called *null plane* of $\mathsf{z}\mathbb{R}$. The point $\mathsf{z}\mathbb{R} = (0, \mathbf{c})\mathbb{R}$ is mapped onto $\mathbb{R}\mathsf{u}' = \mathbb{R}(1, \mathbf{o})$ which is the plane at infinity $x_0 = 0$. All other points at infinity are mapped to planes parallel to the axis A of the helical motion. Using the notation $\mathbf{c} = (c_1, c_2, c_3)$ and $\bar{\mathbf{c}} = (c_4, c_5, c_6)$ one can form the skew symmetric matrix

$$C = \begin{pmatrix} 0 & c_4 & c_5 & c_6 \\ -c_4 & 0 & c_3 & -c_2 \\ -c_5 & -c_3 & 0 & c_1 \\ -c_6 & c_2 & -c_1 & 0 \end{pmatrix}. \qquad (27)$$

It satisfies $C^T = -C$, and the above defined null polarity $\varkappa$ is given by the equation $\mathsf{u}' = C \cdot \mathsf{z}$, which again shows the linearity of $\varkappa$. Note that for any skew symmetric matrix one obtains

$$\mathsf{u}'^T \cdot \mathsf{z} = -\mathsf{z}^T \cdot C \cdot \mathsf{z} = -\mathsf{z}^T \cdot \mathsf{u}' = 0.$$

This expresses the incidence of a point $\mathsf{z}\mathbb{R}$ with its null plane $\mathbb{R}\mathsf{u}'$; point $\mathsf{z}\mathbb{R}$ is called *null point* of that plane.

Any correlation with a regular skew–symmetric transformation matrix is called a null polarity. Our derivations show that any null polarity can be interpreted as mapping from points to path normal planes of a helical motion as long as we map only points which are not at infinity.

Example: A null polarity shall be defined by

$$(x_0, \ldots, x_3) \mapsto (u'_0, \ldots, u'_3) = (-x_3, -x_2, x_1, x_0).$$

Interpreting this in E^3 we have the helical motion defined by $\mathbf{c} = (0,0,1)$, $\bar{\mathbf{c}} = (0,0,1)$ with pitch $p = 1$ and axis

$$(\mathbf{a}, \bar{\mathbf{a}}) = (0,0,1,0,0,0)\mathbb{R},$$

which is the z–axis of a Cartesian coordinate system. The angular velocity is normalized to 1. The velocity vector field reads as $\mathbf{v}(\mathbf{x}) = (-y, x, 1)$, where $(x_0, \mathbf{x})\mathbb{R} = (1, x, y, z)\mathbb{R}$.

The inverse mapping to $\varkappa$ can be derived in the following way. Each plane $\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, \mathbf{u})$ in E^3 not parallel to the axis of a helical motion can be considered as path normal plane of a point $\mathsf{z}\mathbb{R} = (1, \mathbf{z})\mathbb{R}$ not at infinity. The linear dependency of $\mathbf{u}$ and the velocity vector of $\mathsf{z}\mathbb{R}$ can be expressed by $(\bar{\mathbf{c}} + \mathbf{c} \times \mathbf{z}) \times \mathbf{u} = 0$. Since the incidence condition $u_0 = -\mathbf{u} \cdot \mathbf{z}$ holds, one obtains

$$\mathbf{z} = \frac{1}{\mathbf{u} \cdot \mathbf{c}}(-u_0\mathbf{c} + \mathbf{u} \times \bar{\mathbf{c}}).$$

The homogeneous representation of the inverse mapping to $\varkappa$ is then given by

$$\mathbb{R}\mathsf{u} \mapsto \mathsf{z}'\mathbb{R} = (\mathbf{u} \cdot \mathbf{c}, -u_0\mathbf{c} + \mathbf{u} \times \bar{\mathbf{c}})\mathbb{R}. \tag{28}$$

The planes $\mathbb{R}\mathsf{u}$ parallel to the axis , i.e. $\mathbf{u} \cdot \mathbf{c} = 0$, cannot occur as path normal planes and the formula gives image points at infinity. If C is a regular matrix, which defines a helical motion in E^3, the corresponding null polarity is called *regular*.

Let now C be singular, that is $\det(C) = (\mathbf{c} \cdot \bar{\mathbf{c}})^2 = 0$. We exclude rank(C) = 0 and the only interesting case is rank(C) = 2, where C defines a *singular null polarity* with $A = (\mathbf{c}, \bar{\mathbf{c}})$ as singular line or axis.

Comparing formulae (8) and (26), we see that in the singular case the null plane of a point $\mathsf{z}\mathbb{R}$ is the connecting plane with the axis A and is not defined for points on A. Further, comparing formulae (7) and (28) says that the null point of a plane is the intersection point with the axis A and not defined for planes through A.

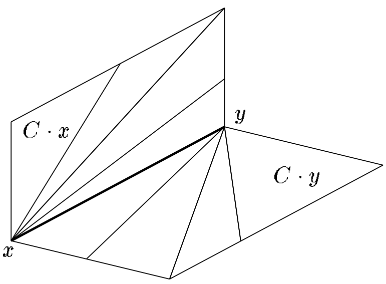

Figure 10: Null lines of a null polarity

Example: The first example for a singular null polarity is

$$(x_0, \ldots, x_3) \mapsto (u'_0, \ldots, u'_3) = (0, -x_2, x_1, 0).$$

In E^3 this corresponds to the rotation around the z–axis with angular velocity 1. The second example, which is derived from a translation parallel to the z–axis, is

$$(x_0, \ldots, x_3) \mapsto (u'_0, \ldots, u'_3) = (-x_3, 0, 0, x_0).$$

But in projective geometry these two examples are equivalent, which means that there exists a linear (projective) mapping, which maps the singular line of the first example (z–axis) to the singular line of the second one (line at infinity of planes orthogonal to the z–axis).

Remark: All regular linear complexes in P^3 are visualizable as path normal complexes of corresponding helical motions in E^3. All singular linear complexes in P^3 are visualizable in E^3 in two ways. Choosing the singular line at infinity leads to a path normal complex of a translation whereas a singular complex with axis A not at infinity can be considered as path normal complex of a rotation around this line A in E^3.

As a correlation, a null polarity maps lines (as point sets) to lines (as carriers of a pencil of planes). Which lines L remain fixed in this line mapping? Consider a point on such a fixed line L; its image plane must pass through the line L. It turns out that this is sufficient: to get a fixed line (also called *null line*), take a point and construct its image plane under the null polarity. Then any line which passes through the point and lies in its image plane is a null line. If C is regular, the path normals of the corresponding helical motion in E^3 are null lines. One can show that the connecting line of two points $\mathsf{x}\mathbb{R}$,

$\mathsf{y}\mathbb{R}$ is a null line exactly if the point $\mathsf{y}\mathbb{R}$ lies in the null plane of the point $\mathsf{x}\mathbb{R}$. Then also $\mathsf{x}\mathbb{R}$ lies in the null plane of $\mathsf{y}\mathbb{R}$, see figure 10.

From the preceding discussion it follows that *the null lines of a null polarity lie in a linear complex and satisfy a linear homogeneous equation in Plücker coordinates.* We have used the following notation

$$\begin{aligned}\Omega(\mathsf{C},\mathsf{L}) &:= \mathbf{l}\cdot\bar{\mathbf{c}} + \bar{\mathbf{l}}\cdot\mathbf{c} \qquad (29)\\ &= c_4 l_1 + c_5 l_2 + c_6 l_3 + c_1 l_4 + c_2 l_5 + c_3 l_6 = 0.\end{aligned}$$

It is now easy to answer the question of the meaning of points in P^5, which do not lie in M_2^4. It will be of importance for the study of rational curves on M_2^4, which are just the Klein images of rational ruled surfaces in P^3, see [23].

The Klein image of a linear complex

With formula (29) it is clear that *the Klein image of the lines of a linear complex $\mathcal{C}$ is a hyperplanar intersection of the Klein quadric M_2^4.* The bilinear form to equation (11) of the Klein quadric shows that the pole of the hyperplane $\Omega(\mathsf{C},\mathsf{X}) = 0$ with respect to M_2^4 is the point $\mathsf{C}\mathbb{R} = (\mathbf{c},\bar{\mathbf{c}})\mathbb{R} \in P^5$. It is called *extended Klein image* of the linear complex $\mathcal{C}$. For the concept of a polarity (special correlation) we refer to any textbook on projective geometry or the book by Boehm and Prautzsch [1].

A singular linear complex $\mathcal{C}$ is characterized by $\Omega(\mathsf{C}) = 0$, which implies that the pole $\mathsf{C}\mathbb{R}$ lies in M_2^4. Thus the hyperplane $\Omega(\mathsf{C},\mathsf{X}) = 0$ is tangent to M_2^4 at that point.

Sets of linear complexes

Sets of linear complexes, whose extended Klein images form a k - dimensional subspace of P^5 are called *k–dimensional spaces of linear complexes.* The intersection of all complexes in such a set is called its *carrier.*

For $k = 1$, we obtain a *pencil of linear complexes.* Let $\mathsf{C}_1\mathbb{R}$ and $\mathsf{C}_2\mathbb{R}$ be the extended Klein images of two independent linear complexes (possibly singular). The Klein images of the lines in these complexes are solutions of

$$\Omega(\mathsf{C}_i,\mathsf{X}) = 0, \Omega(\mathsf{X},\mathsf{X}) = 0, i = 1,2,$$

such that the common lines of both complexes are also lying in any other complex of the pencil, i.e. they satisfy

$$\Omega(\lambda_1\mathsf{C}_1 + \lambda_2\mathsf{C}_2,\mathsf{X}) = 0. \qquad (30)$$

We see that the carrier of the pencil is the intersection of M_2^4 with a 3–space (intersection of the hyperplanes $\Omega(\mathsf{C}_1,\mathsf{X}) = 0$ and $\Omega(\mathsf{C}_2,\mathsf{X}) = 0$), which is the polar space to the line $(\lambda_1\mathsf{C}_1 + \lambda_2\mathsf{C}_2)\mathbb{R}$ with respect to M_2^4. The carrier is in general a *linear congruence* or *net* of lines. This is a two–parameter set of lines which is of one of the following types.

A *hyperbolic net* consists of all lines intersecting two real skew lines which are called *focal lines.* The Klein images of the focal lines are the real intersection points of $(\lambda_1\mathsf{C}_1+\lambda_2\mathsf{C}_2)\mathbb{R}$ with the Klein quadric M_2^4.

An *elliptic net* is formed by the two–parameter set of real lines intersecting two conjugate complex lines. Their Klein images are conjugate complex intersection points of $(\lambda_1\mathsf{C}_1 + \lambda_2\mathsf{C}_2)\mathbb{R}$ with M_2^4. To get a better understanding of elliptic nets we consider a special example in E^3. Take a helical motion with axis A and pitch $p \neq 0$ and let π be a plane orthogonal to A. Then, the tangent lines to trajectories (helices) at all points of the plane π form a special elliptic net, called an *elliptic net of revolution.* These path tangents are generator lines of a one parameter family of one–sheeted hyperboloids of revolution, which possess same axis A, same center $\in A$ and same minor vertices, see figure 11. Applying affine maps to an elliptic net of revolution, one obtains all elliptic nets in E^3.

A *parabolic net* is a limit case between these two, where the line $(\lambda_1\mathsf{C}_1 + \lambda_2\mathsf{C}_2)\mathbb{R}$ is tangent to M_2^4. It may be visualized as set of surface tangents at all points of a ruling of a ruled quadric.

Remark: Since the Klein image of a net is the intersection of M_2^4 with a 3–space, it is a ruled quadric in case of a hyperbolic net. A hyperbolic net constists of two 1–parameter families of pencils of lines, which is reflected in the geometry of the ruled quadric.

The Klein image of an elliptic net is an oval quadric and a parabolic net is mapped under γ onto a quadratic cone.

Visualizing a parabolic net as set of surface tangents as above, the pencils of tangent lines are mapped onto

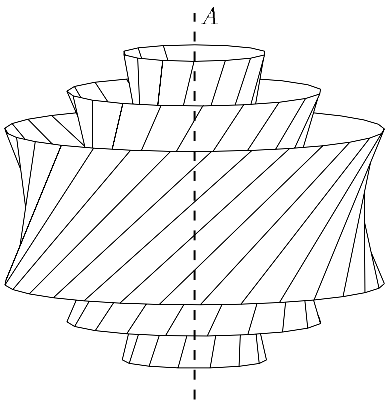

Figure 11: Elliptic net of revolution

the generators of the net's Klein image cone and the γ–image of the generator of the ruled surface is the vertex of the cone.

The case $k = 2$ leads to *bundles of linear complexes*, whose carrier is in general a (not necessarily real) *regulus* (one family of rulings on a ruled quadric).

Let $\mathsf{C}_1\mathbb{R}, \mathsf{C}_2\mathbb{R}$ and $\mathsf{C}_3\mathbb{R}$ be the extended Klein images of three independent linear complexes (possibly singular). The Klein images of the lines in these complexes are solutions of

$$\Omega(\mathsf{C}_i, \mathsf{X}) = 0, \Omega(\mathsf{X}, \mathsf{X}) = 0, i = 1, 2, 3$$

such that the common lines also lie in any of the complexes

$$\Omega(\lambda_1\mathsf{C}_1 + \lambda_2\mathsf{C}_2 + \lambda_3\mathsf{C}_3, \mathsf{X}) = 0. \qquad (31)$$

Thus, the carrier of the bundle is the intersection of M_2^4 with a 2–plane ϵ (intersection of the three hyperplanes $\Omega(\mathsf{C}_i, \mathsf{X}) = 0, i = 1, 2, 3$), which is the polar space to the 2–plane

$$\phi : (\lambda_1\mathsf{C}_1 + \lambda_2\mathsf{C}_2 + \lambda_3\mathsf{C}_3)\mathbb{R}$$

with respect to M_2^4.

In case where ϵ intersect M_2^4 in a conic, the carrier of the bundle of complexes is a (not necessarily real) *regulus*, that is one family of generating lines of a ruled quadric Q. Then, also $\phi \cap M_2^4$ is a conic and its points are the Klein images of the other family of generating lines on Q.

The other types of bundles of complexes can be found by discussing the intersection $\epsilon \cap M_2^4$, which can be a pair of lines, a double counted line or the whole plane ϵ. Here we refer to the literature [8, 36].

Conclusion and outlook

We presented an introduction to line geometry along with applications in constraint solving, reverse engineering and robot kinematics. For that, we enriched the classical material by computational aspects such as approximation by linear complexes. There is a variety of open problems in this area, which would deserve a careful investigation. Let us briefly indicate some directions.

We touched approximation in line space, for which general concepts as well as refined techniques for specific applications need to be developed. Related to that are the numerical aspects of line geometric computations. Other line families than the ones treated in the present text deserve interest. As a contribution in this direction one may view the computational geometric investigation of ruled surfaces in a paper of the same issue [23], but also there a lot of open problems still remain.

Within algorithmic geometry (often called "computational geometry") there is also a lack of results on basic algorithmic tasks in line space. Algorithmic questions about lines in 3–space arise in various applications including ray tracing in computer graphics, motion planning, placement and assembly problems in robotics and object recognition in computer vision. A paper by Chazelle et al.[4] gives initial results, but remarks at the end: "In general, it appears that most questions about lines in space are still open".

Summarizing, the field of "computational line geometry", as a combination of classical line geometry with new questions arising in context of a computational treatment, is an interesting area for future research, both from an academic and practical point of view.

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References

1. Boehm, W. and Prautzsch, H.: *Geometric Concepts for Geometric Design*, A K Peters, Wellesley, MA, 1994.
2. Bottema, O. and Roth, B.: *Theoretical Kinematics*, Dover Publ., New York, 1990.
3. Brauner, H. and Kickinger, W.: *Baugeometrie 1*, Bauverlag, 1977.
4. Chazelle, B., Edelsbrunner, H., Guibas, L.J., Sharir, M. and Stolfi, J.: Lines in space: combinatorics and algorithms, *Algorithmica* **15**, 428–447, 1996.
5. Chen, H.Y. and Pottmann, H.: Approximation by ruled surfaces, *J. of Computational and Applied Math.*, submitted.
6. Edge, W.L.: *The Theory of Ruled Surfaces*, Cambridge University Press, Cambridge, UK, 1931.
7. Ge, Q. J. and Ravani, B.: On representation and interpolation of line segments for computer aided geometric design, *ASME Design Automation Conf.* , **vol. 96-1**, 191–198, 1994.
8. Hlavaty, V.: *Differential Line Geometry* P. Nordhoff Ltd., Groningen, 1953.
9. Hoffmann, Ch.M. and Vermeer, P.J.: Geometric constraint solving in R^2 and R^3, in: *Computing in Euclidean Geometry*, D.Z. Du and F. Hwang, eds., World Scientific Publishing, 1994.
10. Hoschek, J.: *Liniengeometrie*, Bibliograph. Institut, Zürich, 1971.
11. Hoschek, J. and Lasser, D.: *Fundamentals of Computer Aided Geometric Design*, A. K. Peters, Wellesley, MA, 1993.
12. Hoschek, J. and Schneider, M.: Interpolation and approximation with developable surfaces, in: *Curves and Surfaces with Applications in CAGD*, A. Le Méhauté, C. Rabut and L.L. Schumaker, eds., Vanderbilt University Press, Nashville, 185–202, 1997.
13. Hoschek, J. and Schwanecke, U.: *Interpolation and Approximation with Ruled Surfaces*, preprint, TH Darmstadt, 1998.
14. Hunt, K.H.: *Kinematic geometry of mechanisms*, Clarendon Press, Oxford, 1978.
15. Hunt, K.H.: Special configurations of robot–arms via screw theory, *Robotica* **4**, 171–179, 1986.
16. Husty, M., Karger, A., Sachs, H. and Steinhilper, W.: *Kinematik und Robotik*, Springer, 1997.
17. Klein, F.: Die allgemeine lineare Transformation der Linienkoordinaten, in: *Gesammelte math. Abhandlungen I*, Berlin: Springer 1921.
18. Lee, I.–K.: Curve reconstruction from unorganized points, preprint (1998).
19. Lukács, G., Marshall, A.D. and Martin, R.R.: *Geometric least-squares fitting of spheres, cylinders, cones and tori*, preprint, University of Wales, Cardiff, 1997.
20. Martin, R.R., Stroud, I.A. and Marshall, A.D.: Data reduction for reverse engineering. In: *The Mathematics of Surfaces VII* (Goodman, T.N.T., ed.), Information Geometers, 1997.
21. Merlet, J.-P.: Singular Configurations of Parallel Manipulators and Grassmann Geometry, *Int. Journal of Robotics Research*, **vol. 8**, No. 5, 45–56, 1992.
22. Müller, E. and Krames, J.: *Vorlesungen über Darstellende Geometrie III: Konstruktive Behandlung der Regelflächen*, Deuticke, Leipzig/Wien, 1931.
23. Peternell, M., Pottmann, H. and Ravani, B.: On the Computational Geometry of Ruled Surfaces, *Computer Aided Design*, to appear.
24. Pottmann, H., Lü, W. and Ravani, B.: Rational ruled surfaces and their offsets, *Graphical Models and Image Processing* **58**, 544–552, 1996.
25. Pottmann, H., Peternell, M. and Ravani, B.: Approximation in line space, in: *Advances in Robot Kinematics: Analysis and Control*, J. Lenarčič, and M.L. Husty (eds.), Kluwer, 403–412, 1998.
26. Pottmann, H. and Randrup, T.: Rotational and helical surface reconstruction for reverse engineering, *Computing* **60**, 307–322, 1998.
27. Pottmann, H.: Applications of the dual Bézier representation of rational curves and surfaces, in *Curves and Surfaces II*, P.J. Laurent, A. Le Méhauté and L. Schumaker (eds.), AK Peters, 377–384, 1994.
28. Randrup, T.: Approximation by cylinder surfaces, *Computer Aided Design*, to appear.
29. Ravani, B. and Wang, J.W.: Computer aided design of line constructs, *ASME J. Mechanical Design* **113**, 363–371, 1991.
30. Rousseeuw, P.J. and Leroy, A.M.: *Robust Regression and Outlier Detection*, Wiley, New York, 1987.
31. Sauer, R.: *Differenzengeometrie*, Springer, Berlin/Heidelberg, 1970.
32. Sprott, K. and Ravani, B.: Ruled surfaces, Lie groups and mesh generation, *Proc. ASME Design Engineering Techn. Conf.*, paper Nr. DETC97/DAC-3966, 1997.
33. Varady, T., Martin, R. R. and Cox, J.: Reverse engineering of geometric models – an introduction, *Computer Aided Design* **29**, 255–268, 1997.
34. Weiss, E.A.: *Einführung in die Liniengeometrie und Kinematik*, Teubner, Leipzig/Berlin, 1935.

35. Yang, M. and Lee, E.: NC verification for wire–EDM using an R–map, *Computer Aided Design* **28**, 733–740, 1996.

36. Zindler, K.: *Liniengeometrie mit Anwendungen*, 2 volumes, DeGruyter, Berlin, 1902, 1906.