# On the Computational Geometry of Ruled Surfaces

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The paper presents a brief introduction to classical geometry of ruled surfaces with emphasis on the Klein image and studies aspects which arise in connection with a computational treatment of these surfaces. Since ruled surfaces are one parameter families of lines, one can apply curve theory and algorithms to the Klein image, when handling these surfaces. We study representations of rational ruled surfaces and get efficient algorithms for computation of planar intersections and contour outlines. Further, low degree boundary curves, useful for tensor product representations, are studied and illustrated at hand of several examples. Finally, we show how to compute efficiently low degree rational $G^1$ ruled surfaces.
*Keywords: line geometry, Klein image, ruled surface, Bézier representation, planar intersection, surface approximation*

## Introduction

Ruled surfaces are one of the simplest objects in geometric modeling, since they are generated basically by moving a line in space. They occur in several applications such as NC milling with a cylindrical cutter and wire electrical discharge machining (EDM).

Besides articles by Ravani [13, 28], Pottmann [26] and Hoschek [18] there is a certain lack on efficient algorithms concerning the computational treatment of ruled surfaces. We want to interpret ruled surfaces as one parameter sets of lines. The mathematical tools for this, such as Plücker coordinates and Klein image of the set of lines in 3–space are collected in a separate paper of the same issue [23].

We give a short introduction to the geometry of ruled surfaces in Euclidean and projective 3–space. Then we mainly concentrate on rational ruled surfaces, their control structure in line space (Bézier points) and low degree representations. At last, a construction of rational $G^1$ ruled surfaces, composed of quadrics, is given. Some examples, mainly for low degree ruled surfaces shall illustrate the algorithms. Although not everything is documented by a figure, all methods have been implemented by the authors in the way described here.

## Ruled Surfaces in Euclidean 3–Space

A surface $\Phi$ in real Euclidean 3–space $E^3$ is called a *ruled surface*, if it possesses a parametric representation

$$\mathbf{x}(u,v) = \mathbf{a}(u) + v\mathbf{r}(u), u \in I, v \in \mathbb{R}, \qquad (1)$$

where $\mathbf{a}(u)$ is a regular *directrix curve*, i.e. $\dot{\mathbf{a}}(u) \neq \mathbf{o}$ and $\mathbf{r}(u) \neq \mathbf{o}$ is a vector field. The surface contains a one parameter family of lines, since $\mathbf{x}(u_0, v)$ represents the *generator* or *ruling* $R(u_0)$ for a fixed value $u_0$, see figure 1. Roughly speaking, a ruled surface is generated by moving a line in space. $\Phi$ is called a $C^r$ surface if (1) is a $C^r$ parametrization and it is defined over the parameter domain $I \times \mathbb{R}$.

A representation of $\Phi$ in Plücker coordinates can

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be found by

$$\mathsf{R}(u) = (\mathbf{r}(u), \mathbf{a}(u) \times \mathbf{r}(u)).$$

The tangent plane at a regular surface point is spanned by the partial derivative vectors $\mathbf{x}_u = \dot{\mathbf{a}} + v\dot{\mathbf{r}}$ and $\mathbf{x}_v = \mathbf{r}$. Thus, the surface normal at $\mathbf{x}$ is

$$\mathbf{n}(u,v) = \mathbf{x}_u \times \mathbf{x}_v = \dot{\mathbf{a}}(u) \times \mathbf{r}(u) + v(\dot{\mathbf{r}}(u) \times \mathbf{r}(u)).$$

A generator $R(u_0)$ is called *non–torsal* iff

$$\det(\dot{\mathbf{a}}(u_0), \mathbf{r}(u_0), \dot{\mathbf{r}}(u_0)) \neq 0,$$

which expresses linear independence of $\mathbf{n}_1 := \dot{\mathbf{a}} \times \mathbf{r}$ and $\mathbf{n}_2 := \dot{\mathbf{r}} \times \mathbf{r}$. We fix the parameter value $u_0$ and study the normals along the generator $R(u_0)$. The ruled surface formed by these normals is parametrized by

$$\mathbf{y}(v,w) = \mathbf{a}(u_0) + v\mathbf{r}(u_0) + w(\mathbf{n}_1(u_0) + v\mathbf{n}_2(u_0)).$$

Because of bilinearity in the parameters $v$ and $w$ this parametrization represents a hyperbolic paraboloid. The relation between points $\mathbf{x}(u_0, v)$ on $R(u_0)$ and surface normals $\mathbf{n}(u_0, v)$ is bijective and linear because of

$$\mathbf{a}(u_0) + v\mathbf{r}(u_0) \mapsto \mathbf{n}_1(u_0) + v\mathbf{n}_2(u_0). \qquad (2)$$

We consider all planes through the fixed generator $R(u_0)$ and for easier notation we add the point at infinity to this ruling which is obtained for $v = \infty$. Then above relation says that each plane is tangent plane at exactly one point on $R(u_0)$. This relation is bijective and called *contact projectivity.* The plane with normal $\dot{\mathbf{r}}(u_0) \times \mathbf{r}(u_0)$ is tangent at infinity. In other words, if the point $\mathbf{x}(u_0, v)$ runs along $R(u_0)$, its tangent plane (or normal) turns around the generator, see figure 2. But note that this is only true for non–torsal generators.

A generator $R(u_0)$ is called *torsal* iff

$$\det(\dot{\mathbf{a}}(u_0), \mathbf{r}(u_0), \dot{\mathbf{r}}(u_0)) = 0, \qquad (3)$$

and under the assumption that $\mathbf{x}(u_0, v)$ is a regular surface point this means

$$\operatorname{rank}(\dot{\mathbf{a}}, \mathbf{r}, \dot{\mathbf{r}})(u_0) = 2. \qquad (4)$$

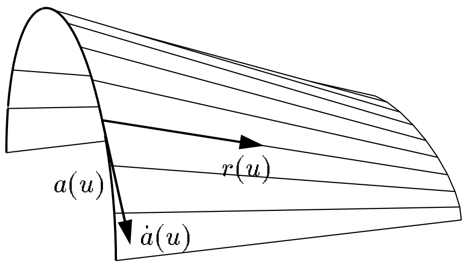


Figure 1: Directrix curve and vector field of a ruled surface

All regular surface points along $R(u_0)$ have a fixed tangent plane.

A special situation occurs if $\operatorname{rank}(\mathbf{r}(u_0), \dot{\mathbf{r}}(u_0)) = 1$. One can easily check that all points on such generators are regular and the surface normal is $\dot{\mathbf{a}} \times \mathbf{r}$. If $\Phi$ is a *cylinder surface* all rulings are of this type and parallel to each other. This motivates that those generators $R(u_0)$ are called *cylindrical.*

We continue the discussion of torsal, non–cylindrical generators. The singular surface points are characterized by linear dependence of $\dot{\mathbf{a}} + v\dot{\mathbf{r}}$ and $\mathbf{r}$. This leads to

$$\dot{\mathbf{a}} \times \mathbf{r} + v\dot{\mathbf{r}} \times \mathbf{r} = \mathbf{o}.$$

A non–cylindrical torsal generator $R(u_0)$ carries exactly one singular point with parameter value

$$v_c = -\frac{(\dot{\mathbf{a}} \times \mathbf{r}) \cdot (\dot{\mathbf{r}} \times \mathbf{r})}{(\dot{\mathbf{r}} \times \mathbf{r}) \cdot (\dot{\mathbf{r}} \times \mathbf{r})}.$$

It is called *cuspidal* point. All other points have $\mathbf{r} \times \dot{\mathbf{r}}$ as surface normal, which is constant along a fixed generator.

If $\operatorname{rank}(\dot{\mathbf{a}}, \mathbf{r}, \dot{\mathbf{r}}) = 2$ holds in a non empty interval and the surface is not cylindrical, $\mathbf{x}(u, v_c(u)) = \mathbf{c}(u)$ is a singular curve on $\Phi$ and called *line of regression* or *cuspidal line*, see figure 3. The ruled surface is called *torsal* or *developable* and $\Phi$ can also be generated as envelope of its one parameter family of tangent planes

$$\mathbf{x} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \det(\mathbf{a}, \mathbf{r}, \dot{\mathbf{r}}).$$

If the curve $\mathbf{c}(u)$ consists of one point only, i.e. $\dot{\mathbf{c}} = \mathbf{o}$, the developable surface is called *cone*.

Although all ruled surface patches in practical use can be embedded in Euclidean 3–space, we will switch to the projective treatment of ruled surfaces, since from the mathematical point of view this representation is more uniform. Most of the properties of ruled surfaces discussed here are invariant under projective transformations, as *torsality* of a ruling, differentiability class of a surface, *order* of an algebraic ruled surface, *rationality* and Bézier representations of rational ruled surfaces.

As we will see in the next section, the geometry of ruled surfaces in projective 3–space $P^3$ is the geometry of curves on the Klein quadric $M_2^4$.

## Ruled Surfaces in $P^3$

Ruled surfaces are one–parameter sets of lines. More precisely, a set $\mathcal{R}$ of lines in $P^3$ is called a $C^r$ *ruled surface*, iff its Klein image $\mathcal{R}\gamma$ is a $C^r$ curve in $M_2^4 \subset P^5$ and therefore possesses a $C^r$ parameterization in Plücker coordinates,

$$u \in I \mapsto \mathsf{R}(u)\mathbb{R} \in P^5. \qquad (5)$$

For local considerations, it is sufficient to consider an open real parameter interval $I$. When discussing algebraic ruled surfaces, we will also admit the real projective line $P^1$ as parameter domain. We will also write $\mathsf{R}\mathbb{R}$ for the Klein image $\mathcal{R}\gamma$ of the ruled surface $\mathcal{R}$. A ruled surface is clearly also a two–parameter set of points in $P^3$. One can prove that it is possible to span each ruling $R(u)$ by two points running on $C^r$ parametric curves

$$\mathsf{a}_0(u)\mathbb{R} \text{ and } \mathsf{a}_1(u)\mathbb{R},$$

which are called *directrix curves*, see figure 2. In Euclidean space we have parametrized each ruling by a parameter $v \in \mathbb{R}$. To obtain also the value $v = \infty$ we will now use a homogeneous real parameter

$$\lambda = (\lambda_1, \lambda_2) \neq (0,0), \text{ i.e. } \lambda \in P^1,$$

to parametrize each ruling. The ruled surface as a point set $\Phi \subset P^3$ is parameterized over $I \times P^1$ via

$$\mathsf{x}(u,\lambda) = \lambda_1 \mathsf{a}_0(u) + \lambda_2 \mathsf{a}_1(u). \qquad (6)$$

Conversely, given this representation, one can define a $C^r$ parameterization in Plücker coordinates

$$\mathsf{R}(u) = \mathsf{a}_0(u) \wedge \mathsf{a}_1(u),$$

and thus the two representations (5) and (6) are equivalent.

A generator $R(u_0)$ of a $C^1$ ruled surface is called *regular* if the Klein image $\mathsf{R}(u_0)\mathbb{R}$ is a regular point of the curve $\mathsf{R}(u)\mathbb{R}$, that means linear independence of $\mathsf{R}(u)$ and $\dot{\mathsf{R}}(u)$. The tangent lines $T(u)$ to the image curve $\mathsf{R}(u)\mathbb{R}$ are given in Plücker coordinates by

$$\mathsf{T}(u) = \alpha\mathsf{R}(u) + \beta\dot{\mathsf{R}}(u), (\alpha,\beta) \neq (0,0). \qquad (7)$$

We want to note that $\dot{\mathsf{R}}(u)$ represents a point on the tangent $T(u)$ in $P^5$.

A very important and powerful concept for further considerations is the order of contact. Recall, that two curves or surfaces are said to have *contact of order* $r$ at a common point $P$, if there exist parameterizations which are regular at $P$ and agree there in all derivatives up to order $r$. Let us now consider two $C^k$ ruled surfaces $\mathcal{R}_1, \mathcal{R}_2$, whose Klein images have *contact of order* $r$ ($r \leq k$) at some point. Then, appropriate parameterizations $\mathsf{R}_1(u), \mathsf{R}_2(u)$ agree at $u = u_0$ up to derivatives of order $r$. In another, equivalent and CAGD related interpretation, we may switch at $u = u_0$ from $\mathsf{R}_1$ to $\mathsf{R}_2$ and obtain a new curve $\mathsf{R}\mathbb{R}$, which has a $C^r$ parameterization $\mathsf{R}(u)$ in Plücker coordinates. Then, the corresponding point representation of the ruled surface $\mathcal{R}$ is $C^r$ and thus the two ruled surfaces $\mathcal{R}_1, \mathcal{R}_2$ possess contact of order $r$ at each regular point of the common generator $R(u_0)$. This is called *contact of order $r$ along the regular generator.*

**Theorem 1** If the Klein image curves $\mathcal{R}_1\gamma, \mathcal{R}_2\gamma$ of two ruled surfaces $\mathcal{R}_1, \mathcal{R}_2$ possess contact of order $r$ at a regular point $R_0\gamma$, then the surfaces $\mathcal{R}_1, \mathcal{R}_2$ possess contact of order $r$ along the common regular generator $R_0$.

In standard CAGD terminology, the theorem says that *$G^r$ joins between curves on the Klein quadric imply $G^r$ joins between ruled surfaces along generators.* We will use this principle for the design and

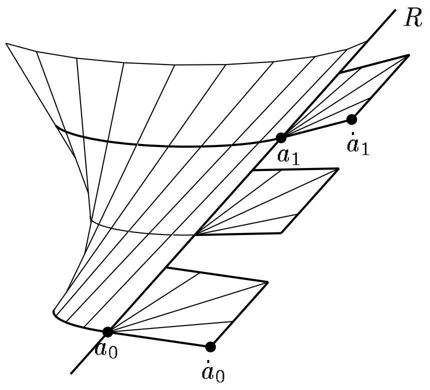


Figure 2: Non–torsal generator and parabolic net of surface tangents

approximation of ruled surfaces. An illustration is given in figure 12.

**Remark:** The converse of this well–known result is not true since it does not work in general for torsal ruled surfaces. We give a counter example: Take two quadratic cones in $P^3$ which are tangent along a common generator, but possess different vertices. The Klein image of the cones are two conics, lying in planes entirely contained in $M_2^4$. Since these planes are Klein images of bundles of lines (with different vertices), these two planes have exactly one point in common. This says that the two conics cannot touch each other.

## First order properties

Let $\mathsf{R}(u)\mathbb{R}$ be a curve on $M_2^4$. It is the Klein image of a ruled surface $\mathcal{R} = R(u)$. First order differential properties of a ruled surface at a generator depend on the tangent line $T$ of its Klein image.

There are two different cases to be distinguished. A regular generator $R(u_0)$ of a $C^1$ ruled surface $\mathcal{R}$ is called *torsal* iff the tangent $T(u_0)$ of the Klein image $\mathcal{R}\gamma$ at $R(u_0)\gamma$ is contained in the Klein quadric $M_2^4$. Using formula (7) this says in particular that

$$\begin{array}{lcl} \mathsf{R}(u_0)\mathbb{R} \in M_2^4 & \text{and} & \dot{\mathsf{R}}(u_0)\mathbb{R} \in M_2^4, \Longleftrightarrow \\ \Omega(\mathsf{R}(u_0)) = 0 & \text{and} & \Omega(\dot{\mathsf{R}}(u_0)) = 0. \end{array}$$

Otherwise, the regular generator is called *non–torsal* and $\dot{\mathsf{R}}(u_0)\mathbb{R}$ is never on $M_2^4$.

We will study the meaning of non–torsal and torsal rulings with respect to a representation by directrix curves $\mathsf{a}_0(u)$ and $\mathsf{a}_1(u)$ in $P^3$. The tangent lines to these curves are parametrized by

$$\begin{array}{rcl} \mathsf{t}_0 & = & \alpha\mathsf{a}_0 + \beta\dot{\mathsf{a}}_0 \\ \mathsf{t}_1 & = & \alpha\mathsf{a}_1 + \beta\dot{\mathsf{a}}_1, (\alpha,\beta) \neq (0,0), \end{array}$$

where $\dot{\mathsf{a}}_0$ and $\dot{\mathsf{a}}_1$ are points and $(\alpha,\beta)$ is a homogeneous parameter on each tangent. A *regular non–torsal generator* $R(u_0)$ is characterized by

$$\operatorname{rank}(\mathsf{a}_0(u_0), \dot{\mathsf{a}}_0(u_0), \mathsf{a}_1(u_0), \dot{\mathsf{a}}_1(u_0)) = 4, \quad (8)$$

which says that the *tangents* of the *directrix curves* $\mathsf{a}_0$ and $\mathsf{a}_1$ are *non–intersecting*. As we already have got to know at the study of ruled surfaces in Euclidean space, all points of a regular generator $R(u_0)$ are regular surface points. Further, the mapping from points of $R(u_0)$ to their tangent planes is the contact projectivity, compare formula (2).

The set of surface tangents at points of $R(u_0)$ is a parabolic net, see figure 2. It consists of pencils of lines with vertices on $R(u_0)$. The extended Klein image of this parabolic net is defined by the tangent line of the curve $\mathsf{R}(u)\mathbb{R}$ at the point $\mathsf{R}(u_0)\mathbb{R}$. That means, the Plücker coordinates $\mathsf{L}$ of surface tangents along $R(u_0)$ satisfy

$$\Omega(\mathsf{R}(u_0), \mathsf{L}) = \Omega(\dot{\mathsf{R}}(u_0), \mathsf{L}) = 0. \quad (9)$$

Compare the section on 'Sets of linear complexes' in the first contribution [23].

The contact projectivity also implies that any plane through a regular non–torsal generator $R(u_0)$ is tangent plane at some point of $R(u_0)$.

Let us move to a *regular torsal generator* $R(u_0)$. Here, the tangent $T(u_0)$ of the Klein image curve lies in the Klein quadric. Its preimage is a pencil of lines in $P^3$, whose vertex and plane are called *cuspidal point* and *torsal plane*, respectively. By the preservation of contact order 1, one can see that the cuspidal point is the only singular point of the ruling, and that all other points have the torsal plane as common tangent plane. Using the point representation (6), a regular torsal ruling is characterized by

$$\operatorname{rank}(\mathsf{a}_0(u_0), \dot{\mathsf{a}}_0(u_0), \mathsf{a}_1(u_0), \dot{\mathsf{a}}_1(u_0)) = 3, \quad (10)$$

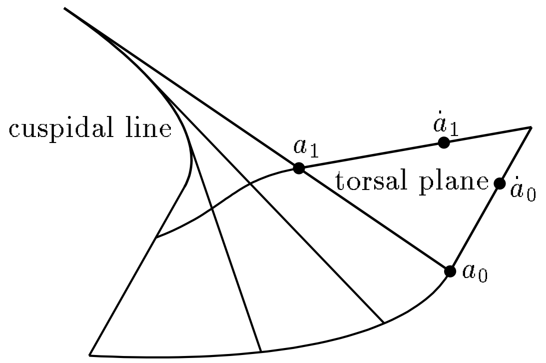


Figure 3: Torsal ruled surface

which expresses *intersecting, but different tangents of the directrix curves.* The plane spanned by these tangents is the torsal plane, see figure 3.

The conversion from a representation in $P^3$ to a representation in Euclidean space can be done as follows. An inhomogeneous Cartesian representation of a ruled surface may be

$$\mathbf{x}(u,v) = (1-v)\mathbf{a}_0(u) + v\mathbf{a}_1(u)$$

with $u \in I$ as before and $v \in \mathbb{R}$. Using the direction vectors $\mathbf{r}(u) = \mathbf{a}_1(u) - \mathbf{a}_0(u)$ of the rulings $R(u)$, we obtain representation (1). Then the just given discussion of torsal and non–torsal rulings leads directly to the classification we had in the last section. But clearly rulings at infinity have to be excluded. Further, in Euclidean space one has to distinguish between non–cylindrical and cylindrical regular torsal generators. The latter possess cuspidal points at infinity. A ruled surface in $P^3$ or $E^3$, all whose rulings are torsal, is called a *torsal ruled surface.* If a ruled surface is not torsal, it is called a *skew ruled surface.*

**Remark:** In any open neighborhood of a generator of a torsal $C^2$ ruled surface $\mathcal{R} \in P^3$, there exists a ruling $R(u_0)$ and an open neighborhood $\mathcal{R}_0 \subset \mathcal{R}$ of $R(u_0)$ such that $\mathcal{R}_0$ is a cone or tangent surface of a $C^1$ curve. Here, the topology in the set of rulings is induced by the topology in the parameter interval. We did not say that the surfaces are piecewise cones (in case of $E^3$, we have to add cylinders, which are cones with vertex at infinity) or tangent surfaces, since the intervals in which the surface type changes may accumulate and cause a problem in such a formulation at the accumulation point. For practical considerations, this situation is not important and thus we model torsal ruled surfaces by appropriately joined pieces of cones and tangent surfaces.

**Remark:** *Developable surfaces* in $E^3$ can be mapped isometrically into the Euclidean plane and are therefore important in applications. It is well–known that these surfaces are pieced together by (parts of) torsal ruled surfaces. Although torsal ruled surfaces will be included in some of our discussions, we will not further elaborate their computational treatment. We just mention that there are two different approaches in the literature [2, 3, 6, 21]: One uses a representation (6) or (1) and ensures developability by fulfilling the condition for torsal generators (10), (3) or (4).

The other approach views the surfaces as envelopes of their one–parameter set of tangent planes, i.e. as curves in dual projective space (see articles [4, 17, 25] and the references therein).

## Algebraic ruled surfaces

A *real irreducible algebraic ruled surface* $\mathcal{R}$ in $P^3$ (using the complex extension) is the Klein preimage of a real irreducible algebraic curve $\mathcal{R}\gamma$ on the Klein quadric $M_2^4 \subset P^5$. The algebraic order of $\mathcal{R}\gamma$ is called the *degree* of $\mathcal{R}$. It can be shown that any irreducible algebraic ruled surface $\mathcal{R}$ in $P^3$ is, as point set, an irreducible algebraic surface $\Phi$.

Recall that the *order* of an algebraic surface $\Phi$ equals the (algebraically counted) number of intersection points with any line $L$ that is not contained in $\Phi$. We will apply this to a ruled surface $\mathcal{R}$. The order of $\mathcal{R}$ equals the (algebr. counted) number of rulings $R(u_i)$ intersecting $L$. If we assume that $\mathsf{R}(u)$ is a proper parametrization of the Klein image $\mathcal{R}\gamma$, this equals the number of zeros of the function

$$\Omega(\mathsf{R}(u), \mathsf{L}).$$

In case of a rational curve $\mathcal{R}\gamma$, this function is just a polynomial. Otherwise, the algebraic ruled surface $\mathcal{R}$ can also be given as intersection of algebraic complexes. This sounds complicated, but the geometric meaning is that the Plücker coordinates $\mathsf{R}$ of the rulings of $\mathcal{R}$ are solutions of a polynomial system of equations; see some simple examples in [23].

Dual to the order of an algebraic surface $\Phi$, the algebraic *class* of $\Phi$ is defined as the number of tan-

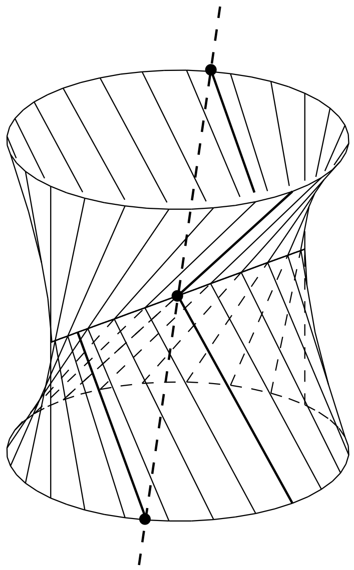

Figure 4: Intersection with a line

gent planes passing through a line $L$ (not in $\Phi$). Applying this to an algebraic skew ruled surface $\mathcal{R}$ we have again to count the number of rulings $R(u_i)$ intersecting $L$, since the contact projectivity (2) guarantees that each plane spanned by $L$ and an intersecting ruling $R(u_i)$ is tangent plane in some point of $R(u_i)$.

But note that the property 'skew' is absolutely necessary here, since in case of a torsal ruled surface $\Phi$, the plane spanned by $L$ and an intersecting ruling will in general not be tangent to $\Phi$.

We summarize that *order and class of a skew irreducible algebraic ruled surface are equal to its degree.* For details see that classical literature [11, 22].

For algebraic ruled surfaces, property 'skew' is much stronger than in the general case. A skew irreducible algebraic ruled surface can have only a finite number of torsal rulings. The torsal rulings $R_0$ are characterized by the property that the tangent line of the Klein image lies on the Klein quadric. In Plücker coordinates, the torsal rulings are calculated by solving

$$\Omega(\dot{\mathsf{R}}, \dot{\mathsf{R}}),$$

which is an algebraic function with only a finite number of zeros. In case of a rational ruled skew surface this can be assumed to be a polynomial.

# Rational ruled surfaces and their Bézier representation

By definition, a real rational ruled surface $\mathcal{R}$ possesses a real rational Klein image curve $\mathcal{R}\gamma$. This curve has a real parameterization $\mathsf{R}(u)\mathbb{R}$, which is polynomial in an inhomogeneous parameter $u$. Hence, we may write the polynomials in the Bernstein basis and obtain the following *Bézier representation of the Klein image*,

$$\mathsf{R}(u) = \sum_{i=0}^{d} B_i^d(u)\mathsf{C}_i. \qquad (11)$$

To parameterize the entire curve one must use a homogeneous parameter running in the real projective line $P^1$. With $u = (u_0 : u_1) \in P^1$, we define

$$B_i^d(u) := \binom{d}{i} u_1^i (u_0 - u_1)^{d-i},$$

and continue to use (11). In case that $u$ is a Lüroth parameter (general curve points belong to exactly one parameter value) and the polynomials in the coordinate functions are relative prime, $d$ equals the degree of $\mathcal{R}$, which is equal to order and class of the point set $\Phi$ of a skew ruled surface. The $d+1$ points $\mathsf{C}_i\mathbb{R} \in P^5$ are Bézier points of $\mathcal{R}\gamma$ and its frame points, see Farin [12], are given by

$$\mathsf{F}_i\mathbb{R} = (\mathsf{C}_i + \mathsf{C}_{i+1})\mathbb{R},\ i = 0, \ldots, d-1.$$

**Remark:** The usual de Casteljau algorithm for polynomial curves in affine space does not require the definition of frame points. This algorithm just uses ratios or affine scales on each leg $\mathbf{b}_i, \mathbf{b}_{i+1}$ of the control polygon $\mathbf{b}_0, \ldots, \mathbf{b}_n$. The affine scale on $\mathbf{b}_i, \mathbf{b}_{i+1}$ is set up by identifying $0 = \mathbf{b}_i$ and $1 = \mathbf{b}_{i+1}$. In case of representing rational curves in projective space one can use weights $w_i, w_{i+1}$ associated to the control points. Instead of the weights one can use frame points on each leg and usually one defines $\mathsf{f}_i\mathbb{R} = (\mathsf{b}_i + \mathsf{b}_{i+1})\mathbb{R}$ to be the projective midpoint of that leg. Since the three points $\mathsf{b}_i\mathbb{R}$, $\mathsf{f}_i\mathbb{R}$ and $\mathsf{b}_{i+1}\mathbb{R}$ define a projective scale (associated with the parameter values $0, 1/2, 1$) on that leg, one can define a projective version of de Casteljau's algorithm, which is appropriate to generate rational curves. By the way, the algorithm based on frame points is invariant under

Figure 5: Control structure of a rational ruled surface and planar intersection

projective transformations, whereas the weights are not invariant.

Apart from $\mathsf{C}_0\mathbb{R}, \mathsf{C}_d\mathbb{R}$, which describe the curve points to parameters $u = 0$ and $u = 1$, these points of the control structure are in general not lying in the Klein quadric $M_2^4$, see figure 5. Hence, they may be viewed as extended Klein images of linear complexes. Any of these complexes uniquely defines a null polarity in $P^3$. We therefore briefly denote $\mathsf{C}_i\mathbb{R}$ and $\mathsf{F}_i\mathbb{R}$ as *Bézier null polarities* and *frame null polarities*; together they form the set of *control null polarities.* They are singular for $\mathsf{C}_0\mathbb{R}$ and $\mathsf{C}_d\mathbb{R}$ and therefore describe rulings for parameter values $u = 0$ and $u = 1$.

**Remark:** There are other methods controlling ruled surfaces in the literature [28, 31].

There are some geometric processing algorithms, as computing planar intersections, contours and contour outlines, in which the line geometric representation and the associated control points of the Klein image turn out to be very useful.

## Planar intersections

First of all we want to discuss the *intersection of the surface $\mathcal{R}$ with a line $L$*: The intersection points are determined by those generators $R(u_i)$ of $\mathcal{R}$ which intersect $L$. The parameter values $u_i$ are obtained by solving

$$\Omega(\mathsf{L}, \mathsf{R}(u)) = 0.$$

The intersection points are $L \cap R(u_i)$. We see that the computation of an implicit representation of the surface can be avoided; for implicitization of rational ruled surfaces we refer to the work by Sederberg and Saito[30].

Let us now intersect a rational ruled surface $\Phi$ given in Plücker coordinates

$$\mathsf{R}(u) = (\mathbf{r}, \bar{\mathbf{r}})(u) = \left(\sum_{i=0}^{d} B_i^d(u)\mathbf{c}_i, \sum_{i=0}^{d} B_i^d(u)\bar{\mathbf{c}}_i\right),$$

with a plane $\mathbb{R}(v_0, \mathbf{v})$ in $P^3$. The control points in $P^5$ are given by $\mathsf{C}_i = (\mathbf{c}_i, \bar{\mathbf{c}}_i)$ and are in general not Plücker coordinates of lines. Intersecting $\mathsf{R}(u)$ with the plane $\mathbb{R}\mathsf{v}$ we use the intersection formula derived in [23]. This leads to

$$\begin{aligned} (p_0, \mathbf{p})(u) &= (\mathbf{v} \cdot \mathbf{r}(u), -v_0\mathbf{r}(u) + \mathbf{v} \times \bar{\mathbf{r}}(u)) \\ &= \sum_{i=0}^{d} B_i^d(u)(\mathbf{v} \cdot \mathbf{c}_i, -v_0\mathbf{c}_i + \mathbf{v} \times \bar{\mathbf{c}}_i). \end{aligned}$$

To obtain a Bézier representation of the rational curve $\mathsf{p}(u)$ we have to define control points

$$\mathsf{d}_i\mathbb{R} := (\mathbf{v} \cdot \mathbf{c}_i, -v_0\mathbf{c}_i + \mathbf{v} \times \bar{\mathbf{c}}_i)\mathbb{R},$$

which can be linearly computed with help of the control points of the Klein image of the ruled surface. Further, the frame points $\mathsf{F}_i\mathbb{R} = (\mathbf{f}_i, \bar{\mathbf{f}}_i)\mathbb{R}$ lead directly to frame points $\mathsf{e}_i\mathbb{R} = (\mathsf{d}_i + \mathsf{d}_{i+1})\mathbb{R}$ of the intersection curve

$$\begin{aligned} &(\mathbf{v} \cdot \mathbf{f}_i, -v_0\mathbf{f}_i + \mathbf{v} \times \bar{\mathbf{f}}_i) = \\ &(\mathbf{v} \cdot \mathbf{c}_i, -v_0\mathbf{c}_i + \mathbf{v} \times \bar{\mathbf{c}}_i) + \\ &(\mathbf{v} \cdot \mathbf{c}_{i+1}, -v_0\mathbf{c}_{i+1} + \mathbf{v} \times \bar{\mathbf{c}}_{i+1}) = \mathsf{d}_i + \mathsf{d}_{i+1} = \mathsf{e}_i. \end{aligned}$$

Summarizing this, we obtain a correctly normalized Bézier representation of the intersection curve

$$(p_0, \mathbf{p})(u) = \sum_{i=0}^{d} B_i^d(u)\mathsf{d}_i. \qquad (12)$$

Representation (12) is linear in the coordinates of the control points $\mathsf{C}_i\mathbb{R}$ of the Klein image of the ruled surface $\mathcal{R}$. Further, we want to give a geometric interpretation of these results. *The control points $\mathsf{d}_i$ and $\mathsf{e}_i$ are null points of the plane $\mathbb{R}\mathsf{v}$ with respect to the control null polarities $\mathsf{C}_i\mathbb{R}$ and $\mathsf{F}_i\mathbb{R}$, respectively.* This leads to the following result, see also [26].

**Theorem 2** Any irreducible planar intersection of a rational ruled surface $\mathcal{R}$ of degree $d$ is representable as a rational Bézier curve of degree $d$. Its Bézier points and frame points are linear in the coordinates of the control points of the Klein image of $\mathcal{R}$.

Parameterization (12) has base points (nontrivial common factors $(u - u_0)$ of the coordinate functions) if the plane $\mathbb{R}\mathsf{v}$ contains a generator $R(u_0)$. Then, one has to divide through the common factor and obtains a degree reduced intersection of the remaining part of the complete intersection. Note that parameterization (12) might be useful without this division if $u_0$ is not in the considered interval; the interpretation of its control points as given in the theorem is still valid, but one has to note that this is a degree elevated representation of part of the complete intersection only.

## Contour of a skew ruled surface

The *contour* of a surface $\Phi$ with respect to a projection center $\mathsf{z}\mathbb{R}$ is the curve of contact of the surface with the tangent cone $\Gamma(\mathsf{z})$ with vertex $\mathsf{z}\mathbb{R}$.

Presenting a surface $\Phi$ in descriptive geometry one uses the *contour outline* with respect to a projection. This is the intersection curve of the tangent cone $\Gamma(\mathsf{z})$ with some chosen image plane $\mathbb{R}\mathsf{v}$, not through $\mathsf{z}\mathbb{R}$. For skew ruled surfaces, represented in Plücker coordinates, there are efficient algorithms to calculate contour and contour outline. The property 'skew' is of importance here, since the contour of a torsal ruled surface mainly consists of those generators whose tangent (torsal) planes pass through the projection center $\mathsf{z}\mathbb{R}$.

Let $\mathsf{R}(u)$ be a Plücker coordinate representation of a ruled surface $\Phi$. For applications in Euclidean space we have to distinguish between parallel and central projection, depending whether the projection center $\mathsf{z}\mathbb{R}$ is at infinity or not. Here we will only discuss the case of central projection and assume that $\mathsf{z}\mathbb{R} = (1, \mathbf{o})\mathbb{R}$ is the origin of our coordinate system.

The generator lines $L$ of the projection cone $\Gamma(\mathsf{z})$ are those surface tangents of $\Phi$, which pass through $\mathsf{z}\mathbb{R}$. Their Plücker coordinates are solutions of

$$\begin{aligned}\Omega(\mathsf{R}(u), \mathsf{L}) = \bar{\mathbf{r}}(u) \cdot \mathbf{l} + \mathbf{r}(u) \cdot \bar{\mathbf{l}} &= 0 \\ \Omega(\dot{\mathsf{R}}(u), \mathsf{L}) = \dot{\bar{\mathbf{r}}}(u) \cdot \mathbf{l} + \dot{\mathbf{r}}(u) \cdot \bar{\mathbf{l}} &= 0 \qquad (13)\\ \bar{\mathbf{l}} &= \mathbf{o}.\end{aligned}$$

For a fixed chosen parameter value $u$, first two equations describe the parabolic net of surface tangents along the generator $R(u)$; the last vector equation guarantees the incidence $\mathsf{z}\mathbb{R} \in L$. From that we easily get a Plücker coordinate representation of the tangent cone

$$\Gamma(\mathsf{z}) : \mathsf{G}(u)\mathbb{R} = (\bar{\mathbf{r}}(u) \times \dot{\bar{\mathbf{r}}}(u), \mathbf{o})\,.$$

To obtain the contour $\mathsf{c}\mathbb{R}$ we just have to intersect the generators $\mathsf{G}\mathbb{R}$ of $\Gamma$ with the generators $\mathsf{R}\mathbb{R}$ of $\Phi$. One can verify that this yields

$$\mathsf{c}(u)\mathbb{R} = (-\mathbf{r} \cdot \dot{\bar{\mathbf{r}}}, \bar{\mathbf{r}}(u) \times \dot{\bar{\mathbf{r}}}(u))\,\mathbb{R}.$$

We want to calculate the contour outline $s$ in some image plane $\mathbb{R}\mathsf{v}$. For that we have to intersect generators $\mathsf{G}(u)\mathbb{R}$ of the tangent cone with the image plane $\mathbb{R}\mathsf{v} = \mathbb{R}(v_0, \mathbf{v})$. This results in a point representation of the contour outline

$$\begin{aligned}\mathsf{s}(u) &= (\mathbf{v} \cdot \mathbf{g}(u), -v_0\mathbf{g}(u) + \mathbf{v} \times \bar{\mathbf{g}}(u)) \\ &= (\mathbf{v} \cdot (\bar{\mathbf{r}}(u) \times \dot{\bar{\mathbf{r}}}(u)), -v_0(\bar{\mathbf{r}}(u) \times \dot{\bar{\mathbf{r}}}(u))) \\ &= (\det(\mathbf{v}, \bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}}), -v_0(\bar{\mathbf{r}}(u) \times \dot{\bar{\mathbf{r}}}(u)))\,. \qquad (14)\end{aligned}$$

If we assume $\mathcal{R}$ to be a rational ruled surface of order $d$, then the tangent cone $\Gamma(\mathsf{z})$ as well as the contour outline $\mathsf{s}(u)\mathbb{R}$ is of order $2(d-1)$. For storing and manipulating data in a CAD-systems, a representation of degree $d$ is possible, if one switches to the dual representation of the tangent cone $\Gamma(\mathsf{z})$. What does this mean?

We start with the interpretation of $\Gamma(\mathsf{z})$ as envelope of its one parameter family of tangent planes $\mathbb{R}\tau(u)$. Since the tangent planes are just the connecting planes of $\mathsf{R}(u)\mathbb{R}$ and $\mathsf{z}\mathbb{R}$, an equation for $\mathbb{R}\tau$ is $\bar{\mathbf{r}}(u) \cdot \mathbf{x} = 0$. Forming homogeneous plane coordinates leads to a dual representation of the tangent cone

$$\Gamma(\mathsf{z})(u) : \tau(u) = \mathbb{R}(0, \bar{\mathbf{r}}(u)). \qquad (15)$$

Further, a representation of the contour outline $\mathsf{s}(u)\mathbb{R}$ as envelope of its tangent lines $T(u)$(dual representation) is then obtained by intersecting tangent

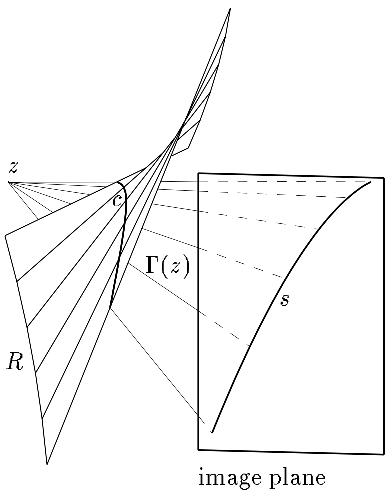


Figure 6: Contour and contour outline of a skew ruled surface

planes $\mathbb{R}\tau(u)$ with the image plane $\mathbb{R}\mathsf{v}$. Plücker coordinates for the tangents $T(u)$ of the contour outline are

$$\mathsf{T}(u) = \mathbb{R}\tau(u) \cap \mathbb{R}\mathsf{v} = (\mathbf{v} \times \bar{\mathbf{r}}, v_0\bar{\mathbf{r}}).$$

The degree for this representation is just $d$. If we want to plot the contour outline, we have to convert to the point representation $\mathsf{s}(u)$. This can be done by intersecting $T(u) \cap \dot{T}(u)$ and we will get back formula (14).

**Remark:** For parallel projection with center $\mathsf{z}\mathbb{R}$ at infinity one cannot use the formulas given here but has to substitute $\bar{\mathbf{l}} = \mathbf{o}$ in (13) by the incidence condition $\mathsf{z}\mathbb{R} \in L$. The calculation of $\mathsf{c}\mathbb{R}$ and $\mathsf{s}\mathbb{R}$ is similar to that one given here.

## Bézier representation of the contour outline

Let $\mathsf{R}(u) = \sum_{i=0}^{d} B_i^d(u)\mathsf{C}_i$ be a Bézier representation of the Klein image of a rational skew ruled surface $\mathcal{R}$, where $\mathsf{C}_i\mathbb{R}$ and $\mathsf{F}_i\mathbb{R} = (\mathsf{C}_i + \mathsf{C}_{i+1})\mathbb{R}$ are control points of the Klein image. A Bézier representation of the contour outline in an image plane $\mathbb{R}\mathsf{v}$ with respect to a central projection with center $\mathsf{z}\mathbb{R} = (1, \mathbf{o})\mathbb{R}$ will be given.

First we start with the dual representation of the tangent cone (compare formula (15))

$$\Gamma(\mathsf{z}) : \tau(u) = \mathbb{R}\left(0, \sum_{i=0}^{d} B_i^d(u)\bar{\mathbf{c}}_i\right),$$

which says that the Bézier and frame planes are

$$\begin{aligned} \gamma_i &= \mathbb{R}(0, \bar{\mathbf{c}}_i), i = 0, \ldots, d, \\ \phi_i &= \mathbb{R}(0, \bar{\mathbf{c}}_i + \bar{\mathbf{c}}_{i+1}), i = 0, \ldots, d-1. \end{aligned}$$

We want to remark that $\gamma_i$ and $\phi_i$ are null planes of the projection center $\mathsf{z}\mathbb{R} = (1, \mathbf{o})\mathbb{R}$ with respect to the Bézier and frame null polarities $\mathsf{C}_i\mathbb{R}$ and $\mathsf{F}_i\mathbb{R}$, respectively.

We proceed as in the last section and intersect tangent planes of $\Gamma(\mathsf{z})$ with the image plane $\mathbb{R}\mathsf{v}$ to get a dual representation of the contour outline in terms of tangent lines $T(u)$. Their Plücker coordinates are

$$\begin{aligned} \mathsf{T}(u) &= (\mathbf{v} \times \sum_{i=0}^{d} B_i^d(u)\bar{\mathbf{c}}_i, v_0 \sum_{i=0}^{d} B_i^d(u)\bar{\mathbf{c}}_i) \\ &= \sum_{i=0}^{d} B_i^d(u)(\mathbf{v} \times \bar{\mathbf{c}}_i, v_0\bar{\mathbf{c}}_i), \\ &= \sum_{i=0}^{d} B_i^d(u)\mathsf{D}_i. \end{aligned}$$

This is a dual Bézier representation [19] of the planar contour outline $s$ and its Bézier lines are $D_i = \mathsf{D}_i\mathbb{R}$. Because of linearity, the frame lines $E_i$ are represented by $(\mathsf{D}_i + \mathsf{D}_{i+1})\mathbb{R}$.

Geometrically this means

$$\begin{aligned} D_i &= \gamma_i \cap \mathbb{R}\mathsf{v}, i = 0, \ldots, d, \\ E_i &= \phi_i \cap \mathbb{R}\mathsf{v}, i = 0, \ldots, d-1, \end{aligned}$$

that these control lines are the intersection lines of the control planes $\gamma_i$ and $\phi_i$ of the tangent cone $\Gamma(\mathsf{z})$ with the image plane $\mathbb{R}\mathsf{v}$.

We want to point out that the dual representation of the tangent cone and the contour outline depends linearly on the control points $\mathsf{C}_i\mathbb{R}$ of the Klein image of $\mathcal{R}$.

We summarize this algorithm: *Compute the control planes $\gamma_i$, $\phi_i$ of the tangent cone $\Gamma(\mathsf{z})$ as null planes of $\mathsf{z}\mathbb{R}$ with respect to the control null polarities $\mathsf{C}_i\mathbb{R}$ and $\mathsf{F}_i\mathbb{R}$ of $\mathcal{R}$.*

*Further, intersecting the control planes $\gamma_i$, $\phi_i$ with the image plane $\mathbb{R}\mathsf{v}$ yields $2d+1$ lines $D_i = \gamma_i \cap \mathbb{R}\mathsf{v}$ and $E_i = \phi_i \cap \mathbb{R}\mathsf{v}$. These are the Bézier and frame lines of the dual Bézier representation of the contour outline.*

If the contour outline is irreducible, the resulting dual parameterization is not degree reducible. For planar intersections we gave an analogous statement. Common factors in the dual parameterization originate from rulings passing through the projection center.

We want to note that the just presented (dual) algorithm computes the contour outline as envelope of projections of its rulings.

Warning: In case of a torsal ruled surface, this is not the contour outline, but the projection of the curve of regression of the torsal ruled surface.

## Low degree representations

Planar intersections are also useful for converting a rational ruled surface given in the Plücker representation (11) into the standard form used in CAGD, namely a tensor product representation. We use two planar intersection curves $\mathsf{a}_0\mathbb{R}$ and $\mathsf{a}_1\mathbb{R}$ in Bézier form as directrix curves,

$$\mathsf{a}_j(u) = \sum_{i=0}^{d} B_i^d(u)\mathsf{b}_{i,j}, \quad j = 0, 1,$$

and obtain the surface as *rational tensor product Bézier surface of degree* $(d, 1)$,

$$\begin{aligned} \mathsf{x}(u,v) &= (1-v)\mathsf{a}_0(u) + v\mathsf{a}_1(u) \\ &= \sum_{i=0}^{d}\sum_{j=0}^{1} B_i^d(u)B_j^1(v)\mathsf{b}_{i,j}. \end{aligned} \tag{16}$$

To get the entire surface, we have to use a homogeneous parameter

$$v = (v_0 : v_1) \in P^1,$$

which gives (16) with $B_0^1 = v_0 - v_1$, $B_1^1 = v_1$. The parameter curves $\mathsf{a}_0\mathbb{R}, \mathsf{a}_1\mathbb{R}$ with Bézier points $\mathsf{b}_{i,j}\mathbb{R}$ are not arbitrarily located. The intersection line of their planes intersects the surface in $d$ points and these $d$ points are intersection points (to common parameter values) of the curves $\mathsf{a}_i\mathbb{R}$.

On the other hand, if a rational ruled surface is given by an arbitrary tensor product representation of degree $(n, 1)$,

$$\mathsf{x}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{1} B_i^n(u)B_j^1(v)\mathsf{b}_{i,j}, \tag{17}$$

the corresponding line coordinate representation is $\mathsf{R}(u) = \mathsf{x}(u,0) \wedge \mathsf{x}(u,1)$,

$$\begin{aligned} \mathsf{R}(u) &= \sum_{k=0}^{2n} B_k^{2n}(u)\mathsf{C}_k, \quad \text{with} \\ \mathsf{C}_k &= \frac{1}{\binom{2n}{k}} \sum_{i+j=k} \binom{n}{i}\binom{n}{j} \mathsf{b}_{i,0} \wedge \mathsf{b}_{j,1}. \end{aligned} \tag{18}$$

Thus, the ruled surface is in general of degree $d = 2n$.

But, as we have said before, each intersection point of the directrix curves $\mathsf{a}_i\mathbb{R} = \mathsf{x}(u,i)\mathbb{R}, i = 0, 1$ to a common parameter value $u_0 \in \mathbb{C}$, leads to a common zero of the 6 coordinate functions. This makes a division by $u - u_0$ possible and reduces the degree of the Plücker coordinate representation by 1. Such a phenomenon occurred above, where we found a $(d, 1)$ tensor product representation (16) for a ruled surface of degree $d$, using directrices $\mathsf{a}_0\mathbb{R}, \mathsf{a}_1\mathbb{R}$ with $d$ common points.

A degree reduction in (18) also happens if the degree of one directrix, say $\mathsf{a}_0$ can be reduced to $m < n$. In other words, we prescribe two directrix curves $\mathsf{a}_0$ and $\mathsf{a}_1$ of degree $m$ and $n$, respectively and write the surface as

$$\begin{aligned} \mathsf{x}(u,v) &= v_0\mathsf{a}_0(u) + v_1\mathsf{a}_1(u) \\ &= v_0\sum_{i=0}^{m} B_i^m(u)\mathsf{b}_{i,0} + v_1\sum_{j=0}^{n} B_j^n(u)\mathsf{b}_{j,1}. \end{aligned} \tag{19}$$

Since this is not a Bézier representation anyway, we have now used another parameter $v = (v_0 : v_1) \in P^1$ on the generators. Then, the line representation is in general of degree $n + m$ and the control points of the Klein image are given by

$$\mathsf{C}_k = \frac{1}{\binom{m+n}{k}} \sum_{i+j=k} \binom{m}{i}\binom{n}{j} \mathsf{b}_{i,0} \wedge \mathsf{b}_{j,1}, \quad k = 0, \ldots, n+m.$$

Here and in the sequel we focus on Bézier representations. It is clear how to modify the results for *piecewise rational ruled surfaces*, given as *B–spline tensor product surfaces* or as *B–spline curves in the Klein quadric*.

There are 'minimal degree representations' of the form (19), since the following result can be proved[11].

**Theorem 3** Any rational ruled surface of degree $d$ in $P^3$ has a representation as a linear blend (19) of a rational directrix curve $\mathsf{a}_0(u)\mathbb{R}$ of degree $m$ and a rational directrix curve $\mathsf{a}_1(u)\mathbb{R}$ of degree $n = d - m \geq m$. These curves do not possess a common point to the same parameter value $u$.

*Proof:* Let us sketch a constructive proof of this well–known result because it is applicable in practical computations. We know of the existence of a representation (19), but need to show $n + m = d$. One may start with planar directrix curves derived from a Plücker coordinate representation with a Lüroth parameter. Thus we can assume that $\mathsf{a}_i(u)$ are Lüroth parameterizations. The intersection curve of the surface and a plane $\mathsf{h} \cdot \mathsf{x} = 0$ is formed by the intersection points between its rulings and the plane, and hence it is parameterized as

$$\mathsf{c}(u) = -(\mathsf{h} \cdot \mathsf{a}_1(u))\mathsf{a}_0(u) + (\mathsf{h} \cdot \mathsf{a}_0(u))\mathsf{a}_1(u).$$

This is a rational curve of order $\leq n + m$. In general, the plane does not contain rulings and thus this curve is the complete intersection. Moreover, we can assume that $\mathsf{c}\mathbb{R}$ is not a singular curve of the ruled surface $\Phi$ of order $d$ (considered as algebraic surface) and thus the order of $\mathsf{c}\mathbb{R}$ equals the order $d$ of $\Phi$. Let us now show that

$$d = n + m - s, \tag{20}$$

where $s$ denotes the algebraically counted number of intersection points of $\mathsf{a}_0\mathbb{R}$ and $\mathsf{a}_1\mathbb{R}$. A reduction in the degree $n+m$ of $\mathsf{c}(u)$ can occur only for a zero $u_0$, i.e.

$$\begin{aligned} \mathsf{c}(u_0) \quad = \quad & -(\mathsf{h} \cdot \mathsf{a}_1(u_0))\mathsf{a}_0(u_0) \\ & +(\mathsf{h} \cdot \mathsf{a}_0(u_0))\mathsf{a}_1(u_0) = 0. \end{aligned} \tag{21}$$

Since the plane does not contain generators it is impossible that both coefficients $\mathsf{h} \cdot \mathsf{a}_i(u_0)$, $i = 0, 1$, vanish simultaneously. Hence a zero $u_0$ implies linearly dependent vectors $\mathsf{a}_i(u_0)$, $i = 0, 1$, which describe an intersection point of the directrix curves $\mathsf{a}_i\mathbb{R}$. This proves (20).

Any such zero $u_0$ implies $\mathsf{a}_1(u_0) = \lambda_0 \mathsf{a}_0(u_0)$ and there might be other zeros, say $u_1, \ldots, u_k$ giving the same relation. Then, consider the new directrix curve

$$\tilde{\mathsf{a}}_1(u) := -\lambda_0 \mathsf{a}_0(u) + \mathsf{a}_1(u).$$

Its parameterization degree can be reduced by the sum $\mu$ of multiplicities of the zero $u_0, \ldots, u_k$, occurring in $\tilde{\mathsf{a}}_1(u)$. Hence, its order equals the maximum of the orders of the directrix curves $\mathsf{a}_0\mathbb{R}, \mathsf{a}_1\mathbb{R}$ minus $\mu$. Simplifying our representation in this way, we end up with directrix curves that do not intersect. Calling their orders again $m \leq n$, we have $d = m + n$. □

If $m < n$, there is exactly one directrix curve of order $m$, since otherwise the degree would be $2m < d$. Directrix curves of order $n$ are

$$\mathsf{c}(u) = v_0(u)\mathsf{a}_0(u) + v_1 \mathsf{a}_1(u), \tag{22}$$

with a polynomial $v_0(u)$ of degree $n - m$ and constant $v_1$. Therefore, we have an $(n - m + 1)$–parameter set of directrices of degree $n$. For $m = n$, there is a one–parameter set of directrices of degree $m$, given by (22) with constant $v_0, v_1$. Examples for rational ruled surfaces of degrees $d = 2, 3, 4$ are given in the next section.

To get a tensor product Bézier representation, the directrix curve with lower degree, say $m$, may be degree elevated to an $n$-th degree representation and then we obtain a tensor product surface of degree $(n, 1)$. This is clearly not the only way to get such a representation, since any two of the $(n - m + 1)$–dimensional set of curves of degree $n$ on the surface may serve as directrices in the Bézier representation.

An interesting application of minimal degree representations is the *construction of low degree trimming curves on rational ruled surfaces.*

Rational cones of degree $d$ are clearly obtained by shrinking one directrix curve to a point $\mathsf{a}_0\mathbb{R}$ $(m = 0)$ and taking another directrix curve $\mathsf{a}_1(u)\mathbb{R}$ of order $n = d$, which does not contain the vertex $\mathsf{a}_0\mathbb{R}$.

## Low degree rational ruled surfaces

Let us exclude cones and briefly discuss the simplest and most important cases of irreducible skew rational ruled surfaces and low degree Bézier representations. We will use that notation of the last section concerning the degrees of the directrix curves.

### Ruled quadrics

Since we have degree $d = 2$ we must have $m = n = 1$ and $\mathsf{a}_0\mathbb{R}$, $\mathsf{a}_1\mathbb{R}$ are linearly parametrized skew lines. The generators connect associated points between $\mathsf{a}_0\mathbb{R}$ and $\mathsf{a}_1\mathbb{R}$, and there is a one–parameter set of such directrix lines. Thus, we find the well–known result that $\mathcal{R}$ is a regulus which possesses a regulus of directing lines. Both reguli lie on the same ruled quadric.

How can this be done practically? Let $L$, $M$ and $N$ be three pairwise skew generators of $\mathcal{R}$. By solving

$$\Omega(\mathsf{L},\mathsf{X}) = \Omega(\mathsf{M},\mathsf{X}) = \Omega(\mathsf{N},\mathsf{X}) = 0, \Omega(\mathsf{X},\mathsf{X}) = 0$$

one gets the regulus $\bar{\mathcal{R}}$ of directrix lines. We choose two independent lines $Y$ and $Z$ from $\bar{\mathcal{R}}$ with linear parametrizations $\mathsf{a}_0(u)$ and $\mathsf{a}_1(v)$, respectively, see figure 7. Intersecting $L$, $M$ and $N$ with $Y$ and $Z$ gives us parameter values $u_i$, $v_i$. The linear blend is then obtained firstly by determining the projective mapping

$$v = \frac{\alpha u + \beta}{\gamma u + \delta}.$$

Inserting $u_i$ and $v_i$ for $i = 1, 2, 3$ here leads to a linear homogeneous system for parameters $\alpha, \dots, \delta$. Inserting this into $\mathsf{a}_1(v)$ gives a parametrization of $\mathcal{R}$

$$\mathsf{x}(u, v) = (1 - v)\mathsf{a}_0(u) + v\mathsf{a}_1(u),$$

with linear $\mathsf{a}_0, \mathsf{a}_1$. Another method computing $\mathcal{R}$ is given in the following.

**Example:** Let again $L$, $M$ and $N$ be three pairwise skew generators of $\mathcal{R}$. Since the Klein image of $\mathcal{R}$ is a conic $C$ on $M_2^4$, a quadratic Bézier representation of $\mathcal{R}\gamma = C$ is

$$\mathsf{C}(t) = (1-t)^2\mathsf{L} + 2t(1-t)w\mathsf{F} + t^2\mathsf{N}, \qquad (23)$$

where $\mathsf{F}$ is the intersection point of tangents to $C$ in $\mathsf{L}\mathbb{R}$ and $\mathsf{N}\mathbb{R}$, and $w$ is a weight guaranteeing that $\mathsf{C}(t)$ is

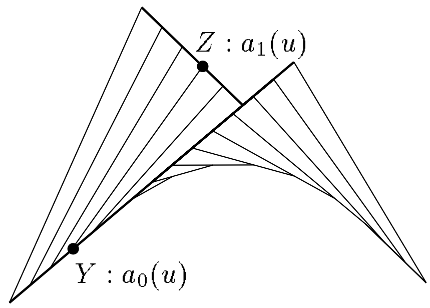


Figure 7: Regulus on a ruled quadric

contained in $M_2^4$. It is easy to verify that $\mathsf{F}$ and $w$ can be chosen as

$$\mathsf{F} = \Omega(\mathsf{N},\mathsf{M})\mathsf{L} - \Omega(\mathsf{L},\mathsf{N})\mathsf{M} + \Omega(\mathsf{L},\mathsf{M})\mathsf{N}, \; w = \pm\sqrt{\frac{-\Omega(\mathsf{L},\mathsf{N})}{2\Omega(\mathsf{F},\mathsf{F})}}.$$

The sign of $w$ determines the arc of $C$ which contains the point $\mathsf{M}$, or the segment of $\mathcal{R}$, which contains the generator $M$.

### Skew ruled cubic surfaces

Before discussing the generation let us note that *any real irreducible skew ruled surface of degree 3 is rational and lies in exactly one linear congruence, which is either a hyperbolic or a parabolic net.* This is easily proved since the Klein image $\mathcal{R}\gamma$ is an irreducible cubic $\subset M_2^4$. It cannot be planar, since planar curves on the Klein quadric are either conics or lie in planes which are entirely contained in $M_2^4$. The latter case corresponds to cones and tangent surfaces of planar curves, which are not skew. Hence, cubic $\mathcal{R}\gamma$ must span a 3–space $G^3$ and therefore it is a rational normal curve and $\mathcal{R}$ is rational. By the way, any irreducible cubic surface, different from a cone, is rational. The intersection of $G^3$ with $M_2^4$ is a quadric $\mathcal{N}\gamma$, Klein image of a net of lines. Since a real cubic cannot lie in an oval quadric, $\mathcal{N}\gamma$ is ruled or a quadratic cone. Hence, $\mathcal{R}$ lies in a hyperbolic or parabolic net $\mathcal{N}$.

By theorem 3 any cubic ruled surface has a unique linearly parameterized directrix line $L = \mathsf{a}_0\mathbb{R}$ corresponding to $m = 1$ and a two–parameter set of directrix conics $\mathsf{a}_1\mathbb{R}$ corresponding to $n = 2$,

$$\mathsf{a}_0(u_0, u_1) \;=\; \mathsf{b}_0 u_0 + \mathsf{b}_1 u_1,$$

$$\mathsf{a}_1(u_0, u_1) \quad = \quad \mathsf{b}_{00}u_0^2 + \mathsf{b}_{01}u_0u_1 + \mathsf{b}_{11}u_1^2,$$

with constant vectors $\mathsf{b}_i$, $\mathsf{b}_{ij}$ and homogeneous parameter $u = (u_0, u_1) \in P^1$. The conics are obtained with formula (22) by inserting

$$v_1 = \text{const. and } v_0 = \alpha_0 u_0 + \alpha_1 u_1.$$

This parameterization is over $P^1 \times P^1$. A parameterization over the projective plane $P^2$ may be obtained by setting $v_1 = 1$. This yields a quadratic homogeneous parameterization in $(u_0, u_1, v_0)$,

$$\mathsf{x}(u_0, u_1, v_0) = v_0\mathsf{a}_0(u_0, u_1) + 1\mathsf{a}_1(u_0, u_1).$$

Therefore the surface has a representation in *triangular quadratic Bézier form.* This parameterization has a base point for parameter values $(u_0, u_1, v_0) = (0, 0, 1)$ that means $\mathsf{x}(0, 0, 1) = \mathbf{o}$.

The straight lines in $P^2$ belong in general to conics on the surface. Those lines $u_0 = ku_1$, which pass through the base point, are mapped to its generators. A condition on the control points of a quadratic Bézier triangle to represent a cubic surface has been derived by W. Degen[9] interpreting the surface as projection of a Veronese manifold in $P^5$ (quadratically parameterized surface $\mathsf{x}(u_0, u_1, u_2) = (u_0^2, u_0u_1, u_0u_2, u_1^2, u_1u_2, u_2^2)\mathbb{R}$).

Let us look at *skew cubic ruled surfaces in a hyperbolic net.* The focal lines of the net are the directrix line $L$ and a line $F$, which is a degenerate directrix conic in form of a quadratically parameterized line. Hence, an easy way to input a cubic ruled surface in a hyperbolic net is the following: Choose a linear parameterization $\mathsf{a}_0(u)$ of a line $L$ and a quadratic parameterization $\mathsf{a}_1(u)$ on a line $F$, skew to $L$, and blend them linearly. It is given by (19) with $m = 1, n = 2$. The three Bézier points and two frame points are five input points for $\mathsf{a}_1(u)$ that can be chosen independently on $F$.

Figure 8 shows a cubic ruled surface with two real torsal generators. The quadratically parametrized focal line is displayed, the other one is at infinity, which means that the generator lines of the surface are parallel to a reference plane.

## Ruled surface through four generators

Let four pairwise skew lines $A_i$, $i = 1, \ldots, 4$ in general position be given. They define a net of lines,

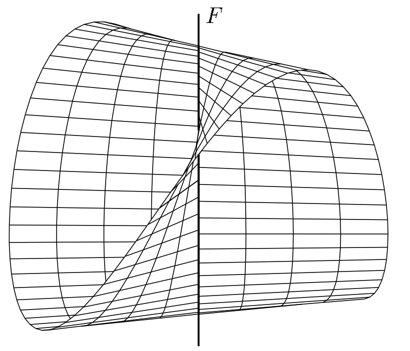


Figure 8: Cubic ruled surface with real torsal lines

which can be hyperbolic, parabolic or elliptic. The Klein images of these four lines $A_i$ span a 3–space $U$ which can be parametrized by

$$U : \mathsf{X} = \sum_{i=1}^{4} \alpha_i \mathsf{A}_i, \alpha_i \in \mathbb{R},$$

and the quadric $Q = U \cap M_2^4$ is the Klein image of all lines in the net defined by $A_i$. This quadric $Q$ can be a ruled quadric, a cone or an oval quadric, as we have discussed earlier [23]. We want to construct a rational ruled surface $\mathcal{R}$ interpolating the given lines and contained in the net. Only the case of a hyperbolic net shall be discussed here.

The axes of the net are those lines $L$ and $F$, which intersect the given lines $A_i$. So their Plücker coordinates are solutions of

$$\Omega(\mathsf{A}_i, \mathsf{X}) = 0, i = 1, \ldots, 4 \text{ and } \Omega(\mathsf{X}, \mathsf{X}) = 0.$$

The first four linear equation are solved by points of a line $g$ polar to $U$ with respect to $M_2^4 \in P^5$. The intersection points

$$g \cap M_2^4 = \{L, F\}$$

also solve the last quadratic equation. These points are real and distinct in case of a hyperbolic net and are the Klein images of the axes.

We choose linear parametrizations $\mathsf{a}_0(u)$ and $\mathsf{a}_1(v)$ on $L$ and $F$, in parameters $u$ and $v$, respectively. The intersection points $L \cap A_i$ and $F \cap A_i$ determine parameter values $u_1, \ldots, u_4$ and $v_1, \ldots, v_4$, respectively. A quadratic parametrization on $F$ can be obtained by using the correspondence

$$v = \frac{f_0 + f_1 u + f_2 u^2}{g_0 + g_1 u + g_2 u^2}. \qquad (24)$$

Each $u$ determines one $v$, that is each point on $L$ determines a corresponding point on $F$. But each point on $F$ corresponds to two points (not necessarily real and distinct) on $L$. Inserting (24) in $\mathsf{a}_1(v)$ leads to a quadratic parametrization on $F$ in $u$.

Practically, we insert the obtained parameter values $u_i$ and $v_i$ in (24) to determine the unknown homogeneous coefficients $f_i, g_i$. This gives four homogeneous linear equations

$$u_i(g_0 + g_1 v_i + g_2 v_i^2) = f_0 + f_1 v_i + f_2 v_i^2, i = 1, \ldots, 4.$$

In general, we have one degree of freedom, which means that there is a one parameter family of cubic skew ruled surfaces interpolating given four lines $A_i$. To obtain a 'unique' solution one can add a linear condition to this linear system, as interpolating a fifth line $A_5$. But $A_5$ has to lie in the net determined by $A_1, \ldots, A_4$. By the way, the solution is just unique up to the choice of $L$ as linear parametrized directrix line.

We summarize: *Five lines of a hyperbolic net can in general be interpolated by two cubic ruled surfaces.*

Looking at the Klein image, this is equivalent to constructing a cubic on a ruled quadric through five given points of the quadric. Note that only 4 arbitrarily located lines in 3–space need not be generators of a cubic ruled surface, since the lines could lie in an elliptic net.

## Rational quartic skew ruled surfaces

A rational skew ruled quartic surface is always contained in a linear complex. One kind of rational quartic ruled surfaces belongs to $m = n = 2$ and carries a one parameter set of conics. The minimal degree parametric representation can be given as a Bézier tensor product surface of degree $(2, 1)$. Among this type of ruled surfaces are also those developable surfaces, which are the tangent surfaces of twisted cubics.

The other type of surfaces to $(m, n) = (1, 3)$ has a unique directrix line and a $n - m + 1 = 3$–parameter set of cubics. Here, a minimal degree representation for the surfaces in $P^3$ is not in tensor product form. A rational Bézier representation is of degree $(3, 1)$, but the parameter curves $v = const.$ have two common points to parameter values $u_0, u_1$. These surfaces, as well as the cubic ruled surfaces discussed before are never developable, since a developable surface with a directrix line needs to be planar.

A refined classification of ruled surfaces of degrees 4,5 and 6 may be found in the classical literature [11, 22].

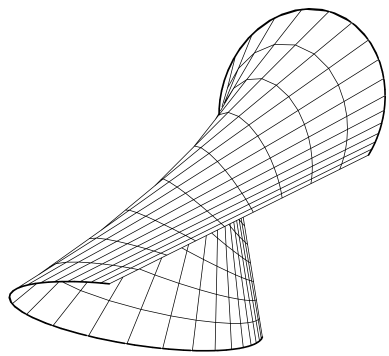

Figure 9: Quartic ruled surface with one parameter set of conics

**Example:** Let a Plücker representation of a quartic skew ruled surface be given

$$\begin{aligned}\mathsf{g}(t) &= \Big(2t(1+t^2), (1+t^2)(t^2-2t+3), \\ &\quad -(1+t^2)(3+t^2), -(3+t^2)^2, 2t(3+t^2), -4t^2\Big).\end{aligned}$$

To apply theorem 3 we intersect the ruled surface with (parallel) planes $\mathsf{v}_0 = (-1, 0, 0, 1)$ and $\mathsf{v}_1 = (1, 0, 0, 1)$ and obtain

$$\begin{aligned}\mathsf{a}_0(t) &= (-(1+t^2)(3+t^2), -4t, \\ &\quad -6-2t-2t^2-2t^3, -(1+t^2)(3+t^2)), \\ \mathsf{a}_1(t) &= (-(1+t^2)(3+t^2), -4t(2+t^2), \\ &\quad -12+2t-10t^2+2t^3-2t^4, (1+t^2)(3+t^2)).\end{aligned}$$

The intersection line of these planes is $(0, 0, 0, 0, 0, 1)\mathbb{R}$ and we obtain the common points of the planar sections for parameter values $t = \pm\mathrm{i}$ and $t = \pm\mathrm{i}\sqrt{3}$. Forming $\mathsf{a}_0 + \mathsf{a}_1$ and $\mathsf{a}_0 - \mathsf{a}_1$ and dividing by a common factor leads to two directrix conics

$$\mathsf{b}_0(t) = (1+t^2, 2t, 3+t^2, 0), \qquad (25)$$

$$\mathsf{b}_1(t) = (0, 2t, t^2-2t+3, -(3+t^2)). \qquad (26)$$

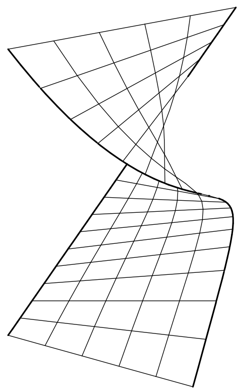

Figure 10: Quartic ruled surface of type $(3, 1)$

Figure 9 shows a patch of the surface. The displayed curves are conics.

**Example:** We proceed as above and start with a Plücker representation

$$\begin{aligned} \mathsf{g}(t) = \quad & \Big((1-t), -t(1+t), -t(t^2-1), \\ & t^3(1+t), t^2(t-1), -2t^2\Big). \end{aligned}$$

We choose planes $\mathsf{v}_0 = (0, -1, 1, 0)$ and $\mathsf{v}_1 = (0, 1, 1, 0)$ and obtain the planar quartics

$$\begin{aligned} \mathsf{a}_0(t) &= (-(1+t^2), -2t^2, -2t^2, -t^2(-1+2t+t^2)), \\ \mathsf{a}_1(t) &= (-(-1+2t+t^2), -2t^2, 2t^2, -t^2(t^2+1)). \end{aligned}$$

These two planes intersect in the line $(0, 0, 1, 0, 0, 0)\mathbb{R}$ and the intersection points correspond to $t = 0$ and $t = \infty$. By adding and subtracting $\mathsf{a}_0$ and $\mathsf{a}_1$ and dividing by a common factor we obtain two cubics, say

$$\begin{aligned} \mathsf{q}_0(t) &= (1+t, 2t, 0, t^2(1+t)), \\ \mathsf{q}_1(t) &= (t-1, 0, -2t^2, -t^2(t-1)), \end{aligned}$$

which have two points in common. Further, $\mathsf{b} = \mathsf{q}_0 + \mathsf{q}_1$ is a linearly parametrized line. With $\mathsf{q}_0$ and $\mathsf{b}$ we can form representation (19). A patch of this type of quartic surface is displayed in figure 10.

## Ruled surface through five generators

Let five pairwise skew lines $A_i$, $i = 1, \ldots, 5$ be given, which span in general a regular linear complex . These lines $A_i$ shall be interpolated by a rational quartic ruled surface $\mathcal{R}$. Any plane in $P^3$ intersects these lines in five points on a conic. So, if a quartic ruled surface exists, it shall be generated as a linear blend between two conics $C_1$, $C_2$, which lie in planes $\pi_1$, $\pi_2$, respectively. Actually, this requires that there exists a projective map $\alpha$ between planes $\pi_1$ and $\pi_2$, which maps $A_i \cap \pi_1$ onto $A_i \cap \pi_2$. We give an algorithm how to find such a map $\alpha$. This is also sufficient, since $\alpha$ restricted to the conics $C_1$, $C_2$ is a projectivity, which means that $\mathcal{R}$ is representable by formula (16) with $d = 2$.

Let $A_i$ be given by their Plücker coordinates $\mathsf{A}_i = (\mathbf{a}_i, \bar{\mathbf{a}}_i)$ and without loss of generality let $\pi_1$ be the plane at infinity $x_0 = 0$. The intersection points $\mathsf{y}_i\mathbb{R} = A_i \cap \pi_1$ are given by $\mathsf{y}_i = (0, \mathbf{a}_i)$. Figure 11 shows a perspective view of this situation. The image points $\mathsf{z}_i\mathbb{R} = (z_{0i}, \mathbf{z}_i)\mathbb{R}$ to $\mathsf{y}_i\mathbb{R}$ under $\alpha$ are given by

$$\begin{pmatrix} z_{0i} \\ \mathbf{z}_i \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{n} \\ \mathbf{o} & M \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mathbf{a}_i \end{pmatrix} \qquad (27)$$

where $\mathbf{n} \in \mathbb{R}^3$ and the regular $3 \times 3$–matrix $M = (m_{ij})$ are unknown. We have to guarantee that the image points $\mathsf{z}_i\mathbb{R}$ are lying on the given lines $A_i$, which can be written as

$$\mathbf{z}_i \cdot \bar{\mathbf{a}}_i = 0, -z_{0i}\bar{\mathbf{a}}_i + \mathbf{z} \times \mathbf{a}_i = \mathbf{o}.$$

These four scalar linear equations are not independent, only two of them are necessary. Inserting (27) into these incident relations gives $5{\times}2$ equations, linear homogeneous in the 12 unknown coefficients of $\mathbf{n}$ and $M$.

The one parameter family of projective mappings $\alpha$ leads to a one parameter family of planes $\pi_2 = \alpha(\pi_1)$ and each plane $\pi_2$ carries a conic $C_2 = \alpha(C_1)$. Let $\mathsf{a}_0(u)$ be a quadratic parametrization of $C_1$. Then $\alpha$ yields a quadratic parametrization of $C_2$, which determines a representation of $\mathcal{R}$ in form (16) with degree $d = 2$.

Corresponding to the one parameter family of projectivities $\alpha$, one would guess that, after free choice of $\pi_1$, there is a one parameter family of quartic ruled surfaces interpolating $A_i$. But actually the one parameter family of conics $C_2 = \alpha(C_1)$ determined above is precisely the one parameter family of conics, which characterize these type of quartic ruled surfaces, see figure 11.

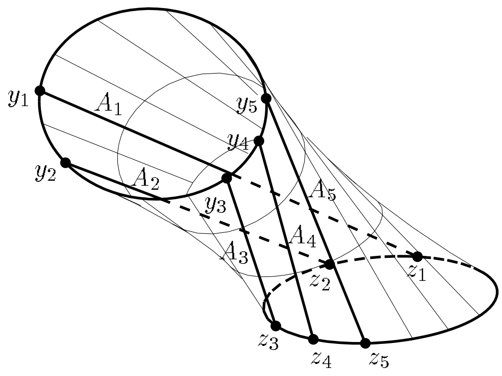


Figure 11: Rational quartic surface through five lines

In other words: *after free choice of $\pi_1$ there is a unique quartic ruled surface interpolating five lines $A_i$.*

## Ruled surfaces in nets

A rational ruled surface $\mathcal{R}$ of degree $d$ all whose generators lie in a line net $\mathcal{N}$ has as Klein image $\mathcal{R}\gamma$ a rational curve of degree $d$ on a quadric $\mathcal{N}\gamma$ in a projective 3–space. Hence, one can establish a one-to-one correspondence between rational curves on quadrics in $P^3$ and rational ruled surfaces in a linear congruence $\mathcal{N}$. This fact, although derived differently, has been used by Dietz, Jüttler and Hoschek [10] in their method for designing rational curves on quadrics (has been extended to surface patches on quadrics as well).

## Approximating ruled surfaces to a set of line segments

For several applications it is appropriate not to work with lines but with line segments. An oriented line segment $\bar{L}$ can be captured by the ordered pair $(\mathbf{p}, \mathbf{q})$ of its endpoints, which naturally defines a mapping to real affine 6–space by $\bar{L} \mapsto (\mathbf{p}, \mathbf{q})$. One can define a useful measure for the distance between two line segments. Approximation in the set of line segments can be translated to curve approximation by using this measurement as distance function in 6–space. This is used by Chen and Pottmann [8] for approximation by ruled surfaces.

If the endpoints $\mathbf{p}, \mathbf{q}$ of the line segments are contained in parallel planes, one can build up an approximation technique which is based on a stereographic projection of the Klein quadric $M_2^4$ and an appropriate distance function in the image 4–space. This will be one of our future work.

# $G^1$ rational ruled surfaces

To construct a $G^1$ ruled surface approximant to a given ruled surface $\mathcal{R}$ one could proceed as follows. Take two planar intersections of $\mathcal{R}$ and approximate them by quadratic rational splines. Blending these quadratic spines linearly using (16) or (19) will produce a piecewise quartic $C^1$ ruled surface, namely a (2,1) tensor product surface.

However, to design a $G^1$ ruled surface without real torsal generators, it is sufficient to use *quadratic ruled surfaces* as segments. If $\mathcal{R}$ possesses real torsal generators, one cannot use segments on quadrics, since they possess no torsal rulings. In this case one has to insert cubic segments. It shall be shown how the Klein image serves for solving this problem by using well–known algorithms for curve design.

A quadratic ruled surface (regulus, quadratic cone or tangent set of a conic) has a conic $\subset M_2^4$ as Klein image. Let us now focus on the case of skew ruled surfaces, since there the line geometric approach is the most elegant one. If two reguli touch each other along a common ruling $R_0$, they possess same contact projectivity and surface tangents, that means same parabolic tangent net along $R_0$, compare formulae (2) and (9).

Hence their Klein images are two conics touching at a common point $R_0\gamma$. We see that *the design of skew, piecewise quadratic $G^1$ ruled surfaces in $P^3$ is equivalent to the design of non–planar quadratic $G^1$ spline curves on the Klein quadric.*

**Remark:** A spline in a plane $\subset M_2^4$ would describe a piecewise quadratic cone or the tangent set of a piecewise quadratic planar curve and therefore refers to torsal ruled surfaces.

## Piecewise quadratic $G^1$ ruled surfaces

We are now proposing an approximation scheme using *conic biarcs on the Klein quadric.* Given a ruled surface without real torsal rulings to be approximated, we select a number of rulings, $R_0, \dots, R_N$ and compute the tangent behaviour (contact projectivity) there. Any pair $R_i, R_{i+1}$ of consecutive rulings plus contact projectivity will be interpolated by a $G^1$ pair $\mathcal{Q}_i, \mathcal{Q}_{i+1}$ of quadratic ruled surfaces.

To construct such a pair $\mathcal{Q}_0, \mathcal{Q}_1$ we look at the Klein image. The input rulings define two points $\mathsf{R}_0\mathbb{R} = R_0\gamma$ and $\mathsf{R}_1\mathbb{R} = R_1\gamma$ on $M_2^4$. The tangent data are mapped to a tangent of $M_2^4$ at each of these points. Additional points on these tangents shall be $\mathsf{T}_0\mathbb{R}$ and $\mathsf{T}_1\mathbb{R}$. They may be computed from a Plücker coordinate representation $\mathsf{R}(u) = \mathsf{a}_0(u) \wedge \mathsf{a}_1(u)$ as

$$\begin{aligned} \mathsf{T}_i = \dot{\mathsf{R}}(u_i) = \\ \dot{\mathsf{a}}_0(u_i) \wedge \mathsf{a}_1(u_i) + \mathsf{a}_0(u_i) \wedge \dot{\mathsf{a}}_1(u_i),\ i = 0, 1. \end{aligned} \tag{28}$$

These two points plus tangents have to be joined by a pair of conic segments $\subset M_2^4$. In the general case, the input data span a 3–space $G^3 \subset P^5$. It intersects the Klein quadric $M_2^4$ in a quadric $\Phi$, on which the conic pair has to lie. Translating to $E^3$, the quadratic ruled surface pair $\mathcal{Q}_0, \mathcal{Q}_1$ lies in a line net, spanned by the input data. We have now reduced our problem to a familiar one, namely the *construction of biarcs on quadrics in 3–space* $G^3$, which was studied by Wang and Joe[32].

Let us briefly describe how the problem in 3–space is solved. Using a projective coordinate system in $G^3$, we have input points $\mathsf{r}_0\mathbb{R}, \mathsf{r}_1\mathbb{R}$ and points $\mathsf{t}_0\mathbb{R}, \mathsf{t}_1\mathbb{R}$ on their tangents. If the quadric $\Phi$ has equation $\mathsf{x}^T \cdot A \cdot \mathsf{x} = 0$, these points satisfy

$$\mathsf{r}_i^T \cdot A \cdot \mathsf{r}_i = 0,\ \mathsf{r}_i^T \cdot A \cdot \mathsf{t}_i = 0,\ i = 0, 1. \tag{29}$$

The Bézier points of the pair of conic segments are $\mathsf{r}_0\mathbb{R}, \mathsf{b}\mathbb{R}, \mathsf{c}\mathbb{R}, \mathsf{d}\mathbb{R}, \mathsf{r}_1\mathbb{R}$. The inner Bézier points of the two segments lie on the given tangents,

$$\mathsf{b} = \mathsf{r}_0 + \lambda\mathsf{t}_0,\ \mathsf{d} = \mathsf{r}_1 + \mu\mathsf{t}_1. \tag{30}$$

Their connection touches $\Phi$ at the point of contact $\mathsf{c}\mathbb{R}$

$$\mathsf{c} = \alpha\mathsf{b} + \beta\mathsf{d},$$

where homogeneous parameters $\alpha$ and $\beta$ have to be such that equation $(\alpha\mathsf{b} + \beta\mathsf{d})^T \cdot A \cdot (\alpha\mathsf{b} + \beta\mathsf{d}) = 0$ has a double root at $\alpha : \beta$. This is equivalent to

$$(\mathsf{b}^T \cdot A \cdot \mathsf{d})^2 - (\mathsf{b}^T \cdot A \cdot \mathsf{b})(\mathsf{d}^T \cdot A \cdot \mathsf{d}) = 0.$$

Inserting (29) and (30), we see that this condition is factored into two independent bilinear relations between $\lambda, \mu$,

$$\mathsf{r}_0^T \cdot A \cdot \mathsf{r}_1 + \lambda\mathsf{t}_0^T \cdot A \cdot \mathsf{r}_1 + \mu\mathsf{r}_0^T \cdot A \cdot \mathsf{t}_1 + \lambda\mu(\mathsf{t}_0^T \cdot A \cdot \mathsf{t}_1 \pm \gamma) = 0. \tag{31}$$

with $\gamma = \sqrt{(\mathsf{t}_0^T \cdot A \cdot \mathsf{t}_0)(\mathsf{t}_1^T \cdot A \cdot \mathsf{t}_1)}$. This says that the tangents (which never lie on the quadric since we have non-torsal rulings $R_i$) have to lie on the same side of the quadric. This is always true for an oval quadric, but needs to be checked otherwise.

If the tangents lie on different sides of a ruled quadric, these data come from a ruled surface which possesses a torsal generator in the considered interval. Such data cannot be approximated by quadric pieces since quadrics do not possess torsal generators. In such cases it is necessary to insert a cubic segment on $M_2^4$ instead of biarcs. How to construct this cubic segment is discussed later.

Note that we have also a direction in which we traverse the set of rulings, which defines the right equation in (31). Because of the bilinearity, the mapping $\mathsf{b}\mathbb{R} \mapsto \mathsf{d}\mathbb{R}$ is a *projective map* and thus *the contact tangents* $\mathsf{b}\mathbb{R}\mathsf{d}\mathbb{R}$ *to the one–parameter set of interpolating biarcs lie on a ruled quadric* $\Gamma$. Quadrics $\Phi$ and $\Gamma$ must be tangent to each other along the curve of possible contact points $\mathsf{c}\mathbb{R}$ of the biarcs. This curve is therefore a *conic*, which passes through $\mathsf{r}_0\mathbb{R}$ and $\mathsf{r}_1\mathbb{R}$. A quadratic parametrization for this conic can be obtained as follows. To represent $\mathsf{c}\mathbb{R}$ we get

$$\mathsf{c} = -(\mathsf{b} \cdot A \cdot \mathsf{d})\mathsf{b} + (\mathsf{b} \cdot A \cdot \mathsf{b})\mathsf{d}.$$

Inserting representations for $\mathsf{b}$ and $\mathsf{d}$ and applying a substitution by using (31) leads to

$$\mathsf{c} = \mp(\mu\gamma)(\mathsf{r}_0 + \lambda\mathsf{t}_0) + \lambda(\mathsf{t}_0 \cdot A \cdot \mathsf{t}_0)(\mathsf{r}_1 + \mu\mathsf{t}_1),$$

after a division by $\lambda$. We still have to express $\mu$ in terms of $\lambda$ by using (31) and get $\mathsf{c}$, quadratically parametrized in $\lambda$. Choosing finally $\lambda$, one can

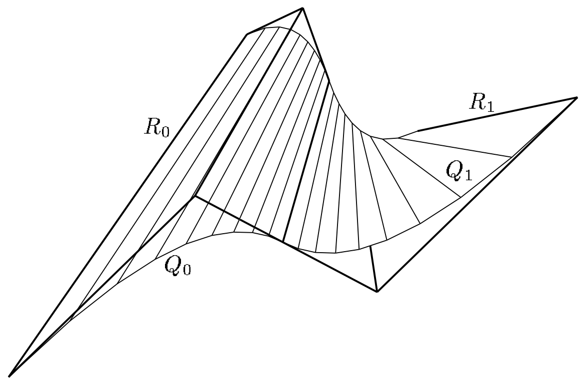


Figure 12: Pair of quadrics interpolating given rulings plus contact projectivity

parametrize the Klein image of a quadratic ruled surface $\mathcal{Q}_i$ similar to (23).

Translating to $P^3$, we see that *there is a one–parameter set of solution pairs* $\mathcal{Q}_0, \mathcal{Q}_1$*; the transition rulings in these pairs form a regulus through* $R_0$ *and* $R_1$.

To pick out a best solution pair $\mathcal{Q}_0, \mathcal{Q}_1$ from this one parameter family one could minimize some useful distance function of the given ruled surface $\mathcal{R}$ and the transition rulings, given by $\mathsf{c}(\lambda)$. Since in practice only segments of rulings actually appear, it is preferable to work with a measure for line segments instead of lines. We do not discuss this here and so the optimal choice of the transition ruling is left.
Figure 12 shows a $G^1$ pair of quadric patches plus control structure. The data come from a cubic surface without real torsal generators.

### Cubic segments

If a ruled surface contains real torsal rulings, we have to insert cubic segments. We consider two generators $R_0, R_1$, let $R_0$ be a torsal and $R_1$ be a non–torsal generator. As before, the Klein image of these data should span a 3–space $G^3$ which intersects $M_2^4$ in a ruled quadric $\Phi$. The case where $G^3$ is tangent to $M_2^4$ such that $\Phi$ is a quadratic cone can be avoided by an appropriate choice of the generator $R_1$. So, let $\Phi$ be a regular ruled quadric with equation $\mathsf{x}^T \cdot A \cdot \mathsf{x} = 0$ and let $\mathsf{r}_0\mathbb{R}, \mathsf{r}_1\mathbb{R}$ and $\mathsf{t}_0\mathbb{R}, \mathsf{t}_1\mathbb{R}$ be given as above. Additionally to relations (29) we have

$$\mathsf{t}_0^T \cdot A \cdot \mathsf{t}_0 = 0, \text{ but } \mathsf{t}_1^T \cdot A \cdot \mathsf{t}_1 \neq 0.$$

The Bézier points of the desired cubic segment are $\mathsf{r}_0\mathbb{R}, \mathsf{b}\mathbb{R}, \mathsf{c}\mathbb{R}$ and $\mathsf{r}_1\mathbb{R}$, with

$$\mathsf{b} = \mathsf{r}_0 + \lambda \mathsf{t}_0, \ \mathsf{c} = \mathsf{r}_1 + \mu \mathsf{t}_1.$$

A cubic segment is tangent to $\mathsf{r}_0\mathbb{R}\mathsf{t}_0\mathbb{R}$ in $\mathsf{r}_0\mathbb{R}$ if and only if the osculating plane $\sigma$ of the cubic at $\mathsf{r}_0\mathbb{R}$ is tangent to $\Phi$ in $\mathsf{r}_0\mathbb{R}$. This implies that $\mathsf{c}\mathbb{R}$ is the intersection point of the tangent line $\mathsf{r}_1\mathbb{R}\mathsf{t}_1\mathbb{R}$ with $\sigma$. One obtains a one parameter family of cubic segments which interpolate the given data.

Let us summarize: Given a ruled surface to be approximated by low degree rational ruled surfaces, we compute a sequence of generators $R_0, \ldots, R_n$, which contains all torsal generators. Each pair $R_i, R_{i+1}$ should contain only one torsal generator. Then one constructs a piecewise $G^1$ ruled surface, which is composed of pairs of quadric pieces if no torsal generators are involved in the considered segment. If a segment determined by two generators $R_i, R_{i+1}$ contains a torsal generator, we insert a cubic piece.

## Conclusion and future research

We have presented some results on the computational treatment of ruled surfaces using line geometry, mainly from the projective point of view. There is still a lot of room for future investigations, some of which we will briefly indicate.

In the study of *ruled surfaces*, algorithms for $G^2$ surfaces of low degree and the inclusion of *Euclidean invariants*[15] are missing. Particularly, parameter of distribution and striction curve should be considered in designing ruled surfaces. Related to that and to so–called Minding isometries between ruled surfaces is a study of 'nearly developable surfaces' and their approximate development (a contribution in a similar direction has been made by G. Aumann[1]). Another topics are the numerical estimation of invariants in connection with results from *difference geometry* [29] and an analysis of line–based *subdivision schemes*.

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