On Approximation in Spaces of Geometric Objects

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Summary. We present a concept for approximation in spaces of geometric objects. It is worked out in detail for approximation problems in the spaces of planes, lines and spheres. Applications include geometric computing with developable surfaces, ruled surfaces and canal surfaces.

Keywords: approximation, duality, line geometry, sphere geometry, developable surface, ruled surface, canal surface

1 Introduction

In advanced classical geometry, *point models* for spaces of simple geometric objects, such as planes, lines and spheres, play an important role. For example, the set of hyperplanes of a projective space is the point set of a projective space (the so-called dual projective space). To give a further example, we note that an appropriate point model for the set of lines in projective 3-space is the so-called Klein quadric Ω in projective 5-space P^5: Lines in P^3 are represented as points of Ω, pencils of lines in P^3 are seen as lines in Ω, ruled surfaces are visualized as curves in Ω, and so on [9].

These classical concepts have recently found a variety of applications in geometric computing. It is, for example, convenient to view a developable surface as envelope of a one-parameter family of planes and thus treat it as a curve in dual projective space [1,2,12,13,23,27]. There are also advantages of viewing a canal surface, which is defined as envelope of a one-parameter family of spheres, as a curve in $\mathbb{R}^4$ (as a point model for the set of spheres) [15–19,24]. Analogously, ruled surfaces can be treated as curves in the Klein quadric [9,20,26,28].

For computational applications the treatment of approximation problems is fundamental. Therefore, we will first discuss how to formulate approximation problems in the sets of spheres, planes and lines and apply these to canal surfaces, developable surfaces and ruled surfaces, respectively. In the final section, we will then summarize by formulating a more general strategy. Thereby, we are able to view the previously discussed cases (spheres, planes, lines) as a special case for approximation in a set of affinely equivalent geometric objects. Connections to kinematic mappings and motion design, and pointers to future research conclude our paper.

2 Approximation in the space of spheres

Recently, it could be shown that certain problems in geometric computing such as computations with canal surfaces, offsets, and medial axis, can be efficiently solved by using concepts from sphere geometry [4,5,15–19,24]. Thus, we will start our investigation with approximation in the space of spheres.

There are different types of classical sphere geometries. *Möbius geometry* incorporates planes as special case of spheres and will not be pursued here. *Laguerre geometry* works with oriented spheres and includes points as special cases of spheres. It will be (partially) used henceforth. *Lie geometry* subsumes both Laguerre and Möbius geometry as special cases.

An oriented sphere in Euclidean 3-space E^3 shall be given by its center (m_1, m_2, m_3) and its signed radius r. Vanishing radius $r = 0$ characterizes points (as degenerate spheres). $r > 0$ belongs to positively oriented spheres (unit normals pointing outside), $r < 0$ characterizes negatively oriented spheres (unit normals pointing inside). Laguerre geometry then also uses oriented planes as another type of fundamental objects and oriented contact as fundamental relation. This will not be needed in the sequel. We will just need the so-called *cyclographic model* for spheres. This is a point model for the set of oriented spheres obtained by the cyclographic mapping ζ, which maps a sphere S with center (m_1, m_2, m_3) and radius r to a point in $\mathbb{R}^4$ via

$$\zeta : S \mapsto (m_1, m_2, m_3, r) \in \mathbb{R}^4. \tag{1}$$

The cyclographic mapping is a well studied classical subject. For its relation to geometric computing, the reader may consult [15,19,24].

Using ζ we can transform a set of spheres into a set of points. In order to solve approximation problems for spheres, we need to come up with an appropriate *distance measure between spheres* and interpret it in the point model. It is easy to see that the natural distance measure of Laguerre geometry (tangential distance) is not useful for our purpose. Hence, we now present an alternative. It has an additional advantage over the tangential distance: whereas the latter leads to a pseudo-Euclidean metric, our distance measure will result in a Euclidean metric in the image space $\mathbb{R}^4$ of the set of spheres.

Given two oriented spheres A, B with

$$\zeta(A) = (a_1, a_2, a_3, a_4),\ \zeta(B) = (b_1, b_2, b_3, b_4),$$

there exists a unique central similarity σ which maps A onto B. Let the centers of A and B be denoted by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, the center of σ is

$$\mathbf{c} = \frac{1}{a_4 - b_4}(a_4\mathbf{b} - b_4\mathbf{a}).$$

It is at infinity for congruent spheres ($a_4 = b_4$). Let D be the difference vector of the point $x \in A$ and the image point $\sigma(x) \in B$ (see Fig. 1). Note

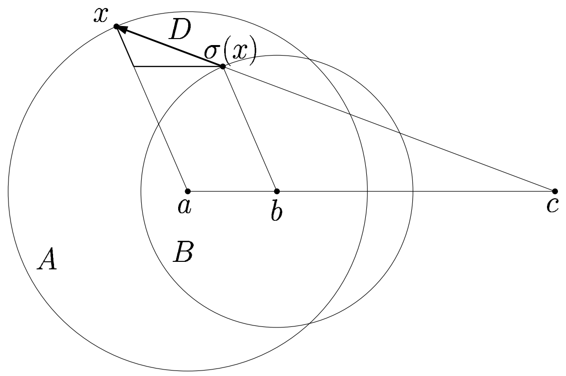

Fig. 1. Distance between two spheres

that all the lines $x\sigma(x)$ pass through $\mathbf{c}$. For concentric spheres, these lines are orthogonal to both spheres, and we then measure orthogonal distances. For other cases, this is not true. However, we can still expect that the integral mean of $\|D\|^2$, viewed as a function on the unit sphere S^2, is a useful measure for the distance $d(A, B)$ of the two spheres,

$$d(A,B)^2 = \frac{1}{4\pi}\int_{S^2} \|D\|^2 d\omega, \tag{2}$$

with $d\omega$ as surface element of S^2.

Surprisingly, this distance measure leads to the canonical Euclidean metric in $\mathbb{R}^4$, since we have the following result.

Theorem 1. *The distance of two oriented spheres A (center $\mathbf{a}$ and radius a_4) and B (center $\mathbf{b}$ and radius b_4), defined via (2), is given by the Euclidean distance of their image points $\zeta(A)$ and $\zeta(B)$ in $\mathbb{R}^4$,*

$$d(A,B)^2 = \sum_{i=1}^{4}(a_i - b_i)^2. \tag{3}$$

Proof. Let the unit sphere S^2 be parametrized by

$$\mathbf{s}(u,v) = (s_1, s_2, s_3)(u,v) = (\cos u \cos v, \sin u \cos v, \sin v).$$

The difference vector $D = x - \sigma(x)$ between corresponding points on A and B and its squared length are given by

$$D(u,v) = \mathbf{a} - \mathbf{b} + (a_4 - b_4)\mathbf{s},$$
$$\|D\|^2 = \|\mathbf{a} - \mathbf{b}\|^2 + (a_4 - b_4)^2 + 2(a_4 - b_4)(\mathbf{a} - \mathbf{b}) \cdot \mathbf{s}(u,v).$$

With the surface element $d\omega = \cos v du dv$ we obtain

$$\int_{S^2} \|D\|^2 d\omega = 4\pi(\|\mathbf{a} - \mathbf{b}\|^2 + (a_4 - b_4)^2) = 4\pi \sum_{i=1}^{4}(a_i - b_i)^2.$$

3 Canal surfaces

A *canal surface* Φ is defined to be the envelope of a one-parameter family of (oriented) spheres

$$S(t) : (x_1 - m_1(t))^2 + (x_2 - m_2(t))^2 + (x_3 - m_3(t))^2 = r(t)^2, \tag{4}$$

where t is a real parameter varying in an interval $[a, b] \subset \mathbb{R}$. The functions $m_i(t)$ and $r(t)$ are considered to be sufficiently often differentiable in $[a, b]$. The center curve (or spine curve) of the spheres shall be denoted by $M(t)$. The envelope Φ is tangent to the spheres $S(t)$ in points of the *characteristic curves* $c(t)$. These curves are obtained by intersecting $S(t)$ with the plane $\dot{S}(t)$,

$$\dot{S}(t) : \sum_{i=1}^{3} (x_i - m_i(t))\dot{m}_i(t) + r(t)\dot{r}(t) = 0. \tag{5}$$

Thus, the family $c(t)$ consists of circles which form one family of principal curvature lines on Φ. The normal vector of $\dot{S}$ is $\dot{M}(t) = (\dot{m}_1, \dot{m}_2, \dot{m}_3)$. The circles $c(t)$ are real (consist of real points) if and only if the reality condition

$$||\dot{M}(t)||^2 - \dot{r}(t)^2 \geq 0 \tag{6}$$

holds. Then, Φ is a real surface.

If equality holds in condition(6) for an isolated parameter value t_0 then the plane $\dot{S}(t_0)$ is tangent to the sphere $S(t_0)$ in an umbilic point of Φ. The sphere $S(t_0)$ has second order contact with Φ there. The canal surface Φ can

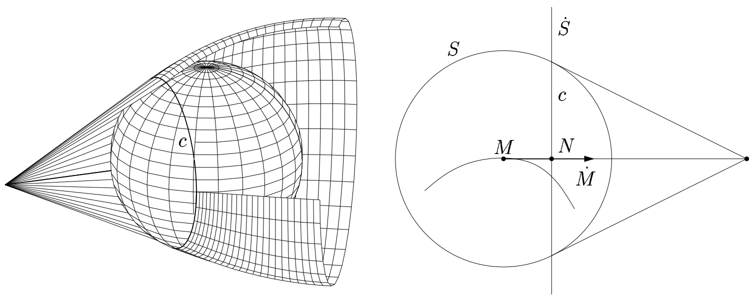

Fig. 2. Geometric properties of a canal surface

be represented by an equation which is obtained by eliminating parameter t from equations (4) and (5). A parametrization of Φ is constructed as follows.

The circles $c(t)$ possess centers $N(t)$ and radii $r_1(t)$ with

$$N(t) = M(t) - \frac{r(t)\dot{r}(t)}{||\dot{M}(t)||^2}\dot{M}(t), \quad r_1(t) = \frac{r(t)}{||\dot{M}(t)||}\sqrt{||\dot{M}(t)||^2 - \dot{r}(t)^2}.$$

The centers $N(t)$ are the intersection points of the tangent lines of $M(t)$ and the planes $\dot{S}(t)$; see Fig. 3. The radii $r(t)$ are obviously real if (6) holds. The curve $M(t)$ possesses an orthonormal frame (T, E, F) with

$$T(t) = \frac{1}{||\dot{M}(t)||}\dot{M}(t), \; E(t) = \frac{1}{||\dot{T}(t)||}\dot{T}(t) \;\text{ and }\; F(t) = T(t) \times E(t),$$

where $X \times Y$ denotes the cross product of two vectors in $\mathbb{R}^3$. T denotes the unit tangent vector of M and E, F form an orthonormal basis of the normal plane. A parametrization of Φ is then obtained by

$$X(t, u) = N(t) + r_1(t)\cos(u)E(t) + r_1(t)\sin(u)F(t). \tag{7}$$

Instead of principal normal vector E and binormal vector F, any other orthonormal basis in the normal plane of the spine curve $M(t)$ may be used in this parameterization.

In particular, let $C(t)$ be an arbitrary piecewise polynomial curve of degree n in $\mathbb{R}^4$. It possesses a B-spline representation,

$$C(t) = \sum_{i=0}^{n} N_i^n(t) C_i, \tag{8}$$

with $N_i^n(t)$ as normalized B-spline functions of degree n over an appropriate knot vector and control points C_i. An analogous representation we have for piecewise rational curves. There, the C_i are homogeneous coordinate vectors of points in the projective extension of $\mathbb{R}^4$, see [22]. It is proved in [18] that the envelope Φ of the one parameter family of spheres $\zeta^{-1}(C(t))$ is a rational surface, thus representable as a rational tensor product spline surface. This result shows that we may use B-spline curves in $\mathbb{R}^4$ in order to compute with canal surfaces, which possess an exact NURBS representation.

Nevertheless, the computation of a rational parametrization is not straightforward: if $m_i(t)$ and $r(t)$ are rational functions, parametrization (7) is in general not rational.

3.1 Approximation of canal surfaces

Approximation algorithms will be based on the interpretation of canal surfaces as envelopes of one-parameter families of spheres $S(t)$. The mapping (1) allows to change the point of view in the sense that we will not consider the envelope Φ in the following but its image curve $\zeta(S(t))$ in the cyclographic model $\mathbb{R}^4$.

Applying the mapping ζ, interpolation or approximation problems concerning spheres and canal surfaces can be transformed into interpolation or approximation problems concerning points and curves in $\mathbb{R}^4$.

Let us consider the following problem. Given k spheres $\Sigma_1, \ldots, \Sigma_k$ and corresponding parameter values $\tau_1, \ldots, \tau_k$, approximate these spheres by a canal surface $\Phi = S(t)$ such that $S(\tau_j)$ is close to Σ_j for $j = 1, \ldots k$. Clearly, close is meant according to the distance (3) between corresponding spheres. We want to determine a canal surface $\Phi = S(t)$ which minimizes

$$F := \sum_{i=1}^{k} d(S(\tau_i), \Sigma_i)^2. \tag{9}$$

If $\zeta(S(t))$ is chosen to be a B-spline curve (8), functional F is quadratic in the coefficients of the unknown control points C_i. Thus, the minimization leads to a linear system.

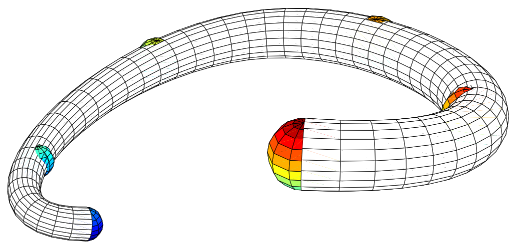

Fig. 3. Canal surface $S(t)$ approximating six spheres; $\zeta(S)$ is a cubic B-spline curve consisting of two segments

Since monotonicity of the radius function is an essential shape property of canal surfaces we will study the following problem. Consider spheres and parameter values as in the previously formulated problem, but additionally a monotonic sequence of radii of the spheres $\Sigma(\tau_i)$. We want to determine an approximating canal surface with monotonic radius function.

A sufficient condition for possessing monotonic radii is that the fourth coordinates c_{i4}, $i = 1, \ldots, k$ of the control points C_i are a monotonic sequence. Thus,

$$c_{04} \leq c_{14} \leq \ldots \leq c_{k-1,4} \leq c_{k4}. \tag{10}$$

Having computed an optimal approximation $X^\star$ by minimizing (9), we will look for an approximation X satisfying constraints (10) and being as close as possible to $X^\star$. This is a constraint optimization problem and is solved by quadratic programming [7].

Since parameter values τ_i have to be chosen in advance and have a great influence on the resulting canal surface one can start with an initial guess and improve it by a parameter correction (see [11]).

4 Approximation in the space of planes

Our motivation for studying approximation in the set of planes comes from the computational geometry of developable surfaces. There, it turned out that viewing these surfaces as envelopes of planes yields computational advantages [1,2,12,13,23,27].

In order to solve approximation problems in the set of planes, it is necessary to introduce an appropriate *distance* between two planes. Euclidean geometry does not directly provide such a distance function. All invariants are expressed in terms of the angle between planes and are inappropriate for our purposes. In view of applications, we are interested in the distances of points of the two planes which are near some region of interest, and this distance can become arbitrarily large with the angle getting arbitrarily close to zero at the same time.

We use the following well-known facts from projective geometry. If we extend real Euclidean 3-space E^3 by ideal points (points at infinity), i.e., intersections of parallel lines, we obtain a model of real projective 3-space P^3. All ideal points form a plane in P^3, the so-called ideal plane. The set of planes in P^3 is a projective space itself, the dual projective space. It is isomorphic to P^3.

Analytically, one uses homogeneous Cartesian coordinates (x_0, x_1, x_2, x_3) for points. For points not at infinity, i.e., $x_0 \neq 0$, the corresponding inhomogeneous Cartesian coordinates will be denoted by

$$x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0}, \quad z = \frac{x_3}{x_0}.$$

A *plane* with equation $u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0$, or, equivalently, $u_0 + u_1x + u_2y + u_3z = 0$ can be represented by its *homogeneous plane coordinates* $U = (u_0, u_1, u_2, u_3)$.

We will later introduce a Euclidean metric in the dual space and thus we first have to obtain the structure of an affine space. Given a projective space P, we obtain an affine space if we remove a hyperplane from P. Thus, we have to remove the points of a plane from the dual space. Viewed from the original space P^3, this means we have to remove a bundle of planes from P^3. Since we actually want to remove the ideal plane, this bundle must have a vertex at infinity. Hence, if we remove all planes passing through a fixed ideal point (for example, planes through the ideal point of the z-axis = planes parallel to the z-axis), we get a set of planes which has the structure of an affine space. This is easily seen in the analytic model. Planes, which are not parallel to the z-axis, can be written in the form

$$z = u_0 + u_1x + u_2y, \tag{11}$$

i.e., they have homogeneous plane coordinates $U = (u_0, u_1, u_2, -1)$. We see that (u_0, u_1, u_2) are affine coordinates in the resulting affine space A^* (of planes, which are not parallel to the z-axis).

We will now introduce a Euclidean metric in A^*. Thereby we make sure that the deviation between two planes shall be measured within some region of interest. This region shall be captured by its projection Γ onto the xy-plane.

For a positive measure μ in $\mathbb{R}^2$ we define the distance d_μ between planes $A = (a_0, a_1, a_2, -1)$ and $B = (b_0, b_1, b_2, -1)$ as

$$d_\mu(A, B) = \|(a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)y\|_{L^2(\mu)}, \tag{12}$$

i.e., the $L^2(\mu)$-distance of the linear functions whose graphs are A and B. This, of course, makes sense only if the linear function which represents the difference between the two planes is in $L^2(\mu)$. We will always assume that the measure μ is such that all linear and quadratic functions possess finite integral.

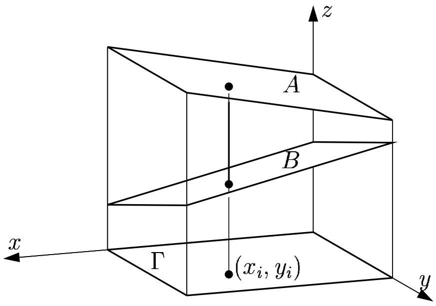

Fig. 4. To the definition of the deviation of two planes

A useful choice for μ is the Lebesgue measure $dxdy$ times the characteristic function χ_Γ of the *region of interest* Γ (Fig. 4). If $\mu = dxdy\chi_\Gamma$, we have

$$d_\mu(A, B)^2 = \int_\Gamma ((a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)y)^2 dxdy. \tag{13}$$

We write $d_\Gamma(A, B)$ instead of $d_\mu(A, B)$. With $c_i := a_i - b_i$, equation (13) can be written as

$$d_\Gamma(A, B)^2 = (c_0, c_1, c_2) \cdot \begin{pmatrix} \int 1 & \int x & \int y \\ \int x & \int x^2 & \int xy \\ \int y & \int xy & \int y^2 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}. \tag{14}$$

This is a quadratic form, whose matrix depends on the domain of integration Γ for the integrals (where we omitted the differentials $dxdy$ for brevity).

Another possibility is that μ equals the sum of several point masses at points (x_j, y_j); see [12]. In this case we have

$$d_\mu(A, B)^2 = \sum_j ((a_0 - b_0) + (a_1 - b_1)x_j + (a_2 - b_2)y_j)^2. \tag{15}$$

Theorem 2. *The distance d_μ defines a Euclidean metric in the set of planes of type (11), if and only if μ is not concentrated in a straight line.*

Proof. See [27].

In this way, approximation problems in the set of planes are transformed into approximation problems in the set of points in Euclidean 3-space, whose metric is based on d_μ. In the next section, we will illustrate this at hand of developable surfaces.

5 Approximation algorithms for developable surfaces

5.1 Developable NURBS surfaces as envelopes of planes

Developable surfaces can be isometrically mapped (*developed*) into the plane, at least locally. When sufficient differentiability is assumed, they are characterized by vanishing Gaussian curvature. A non-flat developable surface is the envelope of its one parameter family of tangent planes. Such a developable surface locally is either a conical surface, a cylindrical surface, or the tangent surface of a twisted curve. Globally, of course, it can be a rather complicated composition of these three surface types. Thus, developable surfaces are ruled surfaces, but with the special property that they possess the same tangent plane at all points of the same generator (=*ruling*).

Because in all points of a generator line the tangent plane is the same, we can identify a developable surface with the one-parameter family of its tangent planes $U(t)$, or in other words, with a certain curve in dual projective space. If this curve is a NURBS curve

$$U(t) = \sum_{i=0}^{n} U_i N_i^k(t), \tag{16}$$

the original surface is a *developable NURBS surface*. Methods for computing a parameterization in standard NURBS tensor product form have been developed [23].

The symbol U_i denotes a homogeneous coordinate quadruple of the i-th *control plane* U_i. Of course the coordinate quadruple contains more information than just the plane as a point set, but for simplicity we just speak of the coordinates of the plane.

For the approximation algorithms discussed in this paper, we will restrict the class of developable surfaces we are working with: We only consider surfaces whose family of tangent planes is of the form

$$U(t) = (u_0(t), u_1(t), u_2(t), -1) \quad \Longleftrightarrow \quad z = u_0(t) + u_1(t)x + u_2(t)y. \tag{17}$$

For NURBS surfaces this is equivalent to the choice of control planes $U_i = (u_{0,i}, u_{1,i}, u_{2,i}, u_{3,i})$ such that always $u_{3,i} = -1$. This means that for all possible planes U we no longer allow to choose an arbitrary coordinate quadruple describing U, but we restrict ourselves to the unique one whose last coordinate equals -1. This is not possible if the last coordinate is zero, so we have to exclude all surfaces with tangent planes parallel to the z-axis. In most cases this requirement is easily fulfilled by choosing an appropriate coordinate system.

Dual projective space with the bundle of planes $(u_0, u_1, u_2, 0)$ removed is an affine space and (u_0, u_1, u_2), describing the plane $(u_0, u_1, u_2, -1)$, are affine coordinates in it. The surfaces (17) become ordinary piecewise polynomial B-spline curves in this dual model.

Recently, algorithms for the computation with the dual representation, the conversion to the standard tensor product representation and the solution of interpolation and some approximation algorithms have been developed [12,13,23]. The general approximation scheme briefly outlined below is discussed in detail in [27]. We include it here since it is a typical example for our concept of geometric approximation.

5.2 Approximation of tangent planes

Consider the following approximation problem. Given m planes $V_1, \ldots, V_m$ and corresponding parameter values v_i, approximate these planes by a developable surface $U(t)$, such that $U(v_i)$ is close to the given plane V_i within an associated area of interest, where i ranges from 1 to m.

The meaning of 'close' is the following: There is a Cartesian coordinate system fixed in space such that all planes are graphs of linear functions of the xy-plane. Its third unit vector may be found as solution of a regression problem to the given plane normals. For all i there is a region of interest Γ_i, or, more generally, a measure μ_i, in the xy-plane. We want to minimize

$$F_1 := \sum_{i=1}^{m} d_{\mu_i}(V_i, U(v_i))^2, \tag{18}$$

for an unknown developable surface $U(t)$. If $U(t)$ is a NURBS surface of type (17), F_1 is a quadratic function in the unknown coordinates of the control planes U_i. These can then be found by solving a linear system of equations.

A good choice for μ_i would be $w_i \chi_{\Gamma_i} dxdy$. An example of this can be seen in Fig. 5. The positive weights w_i can be used to assign more or less importance to the single parameter values v_i. It would also be possible to choose different coordinate systems for different planes V_i, but this is not necessary, because it is equivalent to multiplying the weights w_i with appropriate factors. With $w_i = \sin^2 \gamma_i$, where γ_i is the Euclidean angle, which is enclosed between V_i and the z-axis, we can correct the influence of measuring distances in the z-direction of a fixed coordinate system for all i.

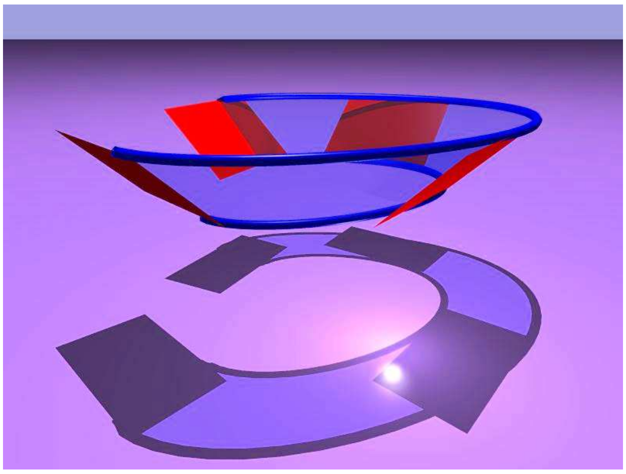

Fig. 5. Approximation of a set of planes by a developable surface

One may fix some boundary control planes in order to ensure a smooth join of subsequent surface segments. Note that the computation of the surface $U(t)$ is equivalent to a polynomial B-spline curve approximation problem using different Euclidean metrics at different points to be approximated. Working with the same μ or Γ for all planes, we get an ordinary curve approximation problem in Euclidean 3-space [6,11,22].

Since the parameters v_i have to be fixed in advance and another choice could have given better results, one will start with an initial guess and then improve it by *parameter correction*. With the Euclidean norms defined above, we can directly apply the known computational schemes [11].

For more details and extensions, such as approximation of points and generators and control of the curve of regression, we refer to Pottmann and Wallner [27].

6 Approximation in line space and applications

Recent research on surface reconstruction, kinematics of parallel manipulators and NC machining [3,25,31] led us to various approximation problems in line space. Thus we started to develop a concept for approximation in line space which will be outlined and illustrated in the sequel.

To understand the nature of the problem, let us briefly review a few facts from line geometry [9,10,26,28].

For two points with homogeneous coordinates $(x_0, \ldots, x_3)$ and $(y_0, \ldots, y_3)$, one defines the six homogeneous *Plücker coordinates* of the spanning line L as

$$(l_1, \ldots, l_6) := (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12}) \text{ with } l_{ij} := x_i y_j - x_j y_i. \tag{19}$$

These coordinates do not depend on the choice of the two points on L and are related by the *Plücker identity*

$$l_1 l_4 + l_2 l_5 + l_3 l_6 = 0. \tag{20}$$

There is a bijective map between ordered, homogeneous 6–tuples $(l_1, \dots, l_6) \neq (0, \dots, 0)$ of real numbers and lines in real projective 3–space P^3. Therefore, one may view the six Plücker coordinates of a line L as homogeneous coordinates of a point $\gamma(L)$ in real projective 5–space P^5. The thereby defined *Klein mapping* γ provides a bijection between the set of lines $\mathcal{L}$ in P^3 and the set of points in a quadric $\Omega \subset P^5$ with equation (20), usually referred to as *Klein quadric.* We see that projective line space, which is clearly a four-dimensional manifold, has the structure of a quadric in P^5.

If we work in $\mathbb{R}^3$ and thereby rule out lines at infinity, we remove a 2–dimensional plane (γ–image of the lines at infinity) from the Klein quadric. Thus, approximation in the set $\mathcal{L}$ of lines in $\mathbb{R}^3$ means approximation in the Klein quadric with a plane Π being removed. Unfortunately the resulting set $\Omega' := \Omega \setminus \Pi$ does not have the structure of an affine space. However, it is well–known that removal of a cut with a tangential hyperplane Γ at some point C of a quadric gives the structure of an affine space. The mapping to an affine space is then realized by *stereographic projection* with center C. Stereographic projection of a quadric Φ from one of its points $C \in \Phi$ is the restriction to the quadric of a central projection with center $C \in \Phi$ and any image hyperplane (not through C). The affine structure in the image hyperplane H is obtained by removing from the projective hyperplane the points of the tangent hyperplane Γ of Φ at C. The reader may visualize this with help of the familiar stereographic projection of a sphere.

From line geometry we know that the points of a tangential cut of Ω at a point $C \in \Omega$ are the Klein images of all lines in 3–space which intersect the line $U = \gamma^{-1}(C)$. Since we work in $\mathbb{R}^3$ and thus neglect lines at infinity, *the following operation introduces an affine structure into $\mathcal{L}$: remove all lines which intersect some line at infinity U.* Let U be the line at infinity of the xy–plane and thus we remove the set $\mathcal{L}'$ of lines orthogonal to the z–axis to get $\mathcal{L}^o := \mathcal{L} \setminus \mathcal{L}'$.

The realization of the corresponding stereographic projection with center C is very simple: A line $A \in \mathcal{L}^o$ may be defined by its intersection point $A_0 = (a_1, a_2, 0)$ with $\pi_0 : \ z = 0$ and its intersection $A_1 = (a_3, a_4, 1)$ with $\pi_1 : \ z = 1$ (see Fig. 6). The mapping

$$\sigma : \ A \in \mathcal{L} \mapsto (a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \tag{21}$$

from the set $\mathcal{L}^o$ onto real affine 4–space describes a *stereographic projection of the Klein quadric.* This is proved as follows. The Plücker coordinates of L and U are

$$\gamma(A) = (a_3 - a_1, a_4 - a_2, 1, a_2, -a_1, a_1 a_4 - a_2 a_3), \ \gamma(U) = (0, \dots, 0, 1).$$

Embedding $\mathbb{R}^4$ into P^5 by

$$\sigma(A) = (a_1, a_2, a_3, a_4) \mapsto \sigma'(A) = (a_3 - a_1, a_4 - a_2, 1, a_2, -a_1, 0),$$

it follows that the point $\sigma'(A)$ is collinear with $\gamma(A)$ and $\gamma(U) = C$, and thus it is the image of $\gamma(A)$ for projection with center C onto the hyperplane $H : x_6 = 0$ in P^5.

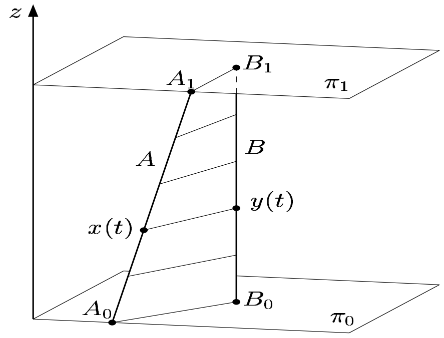

Fig. 6. Introducing a distance measure between two lines in an area of interest located between parallel planes π_0, π_1

For approximation we need a distance measure between two lines A, B. In practice, the distance within some area of interest will be important. Placing this area between the parallel planes π_0 and π_1, we may map the two lines linearly onto each other via

$$x(t) = (1 - t)A_0 + tA_1 \mapsto y(t) = (1 - t)B_0 + tB_1. \tag{22}$$

Viewing the z-axis as vertical, we may say that the line segments $x(t)y(t)$ are horizontal (cf. Fig. 6). It seems reasonable to measure the deviation of the two lines A, B in the domain between the reference planes π_0, π_1 by

$$d(A, B)^2 = \int_0^1 \|x(t) - y(t)\|^2 dt.$$

Inserting (22) we obtain

$$\begin{aligned} d(A, B)^2 &= (A_0 - B_0)^2 + (A_1 - B_1)^2 + (A_0 - B_0) \cdot (A_1 - B_1) \\ &= \sum_{i=1}^{4} (a_i - b_i)^2 + (a_1 - b_1)(a_3 - b_3) + (a_2 - b_2)(a_4 - b_4). \end{aligned} \tag{23}$$

This is the distance of their image points $\sigma(A), \sigma(B) \in \mathbb{R}^4$ in a *Euclidean metric in* $\mathbb{R}^4$, defined by positive definite quadratic form

$$\langle X, X \rangle = x_1^2 + \ldots + x_4^2 + x_1 x_3 + x_2 x_4. \tag{24}$$

For the definition of $d(A, B)$ one integrates squared distances between A, B measured in parallel planes between π_0, π_1. These distances differ from orthogonal distances to A by a factor between 1 and $1/\cos\varphi$, if φ is the angle between A and the z–axis. At least for lines whose angle with the z–axis does not exceed some tolerance dependent value $\gamma_0 < \pi/2$, (23) is a useful distance measure. Its behaviour is a counterpart to the well–known fact, that the distance distortion for stereographic projection of a sphere in $\mathbb{R}^3$ increases with the distance from the antipodal point of the projection center.

Theorem 3. *Consider two parallel planes π_0, π_1 in $\mathbb{R}^3$ and the set $\mathcal{L}^o$ of all lines which are not parallel to them. Then intersection of any line in $\mathcal{L}^o$ with π_0, π_1 gives a pair (p, q) of points, which may be considered as point in real affine 4–space $\mathbb{R}^4$. This mapping from $\mathcal{L}^o$ onto $\mathbb{R}^4$ can be interpreted as stereographic projection of the Klein quadric. The image space $\mathbb{R}^4$ can be endowed with a Euclidean metric (in an adapted coordinate system given by (24)), which corresponds to the deviation of the lines within the parallel strip bounded by planes π_0, π_1.*

This result provides a *transfer principle from approximation in line space to approximation in Euclidean 4–space*. We will illustrate it at hand of some examples.

6.1 Scattered data fitting in line space

We consider the problem of *scattered data interpolation and approximation for functions defined on line space*. Let a finite set of lines L_i (data lines) and associated real numbers f_i, obtained by some measurement or computation, be given. We would like to construct a function F, which is defined on all lines L within some domain D of interest (for example, lines with a maximum distance d to a fixed point) and exactly or approximately satisfies $F(L_i) = f_i$. One approach is the following [21]. Consider a centrally symmetric covering of the unit sphere $\Sigma \in \mathbb{R}^3$ by circular caps Γ_i, $i = 1, \dots, m$ with rotational axes a_i and spherical radii ρ_i. For each axis a_i, let $\mathcal{L}_i$ be the set of lines that form an angle $< \rho_i$ with a_i and lie in the domain of interest. With two parallel planes that are orthogonal to a_i and enclose the domain of interest, we perform the mapping into $\mathbb{R}^4$. There, the images of data lines $L_k \in \mathcal{L}_i$ are data points with associated function values. Using the corresponding metric, they can be interpolated or approximated by any method which works in $\mathbb{R}^4$, for example radial basis functions [11]. Of course, we use the metric based on (24) instead of the canonical Euclidean metric. This gives a partial solution function F_i defined on any line in $\mathcal{L}_i$. Finally one just has to combine these partial solutions into a single one. This approach has been discussed in more detail by Peternell and Pottmann [21].

6.2 Ruled surface approximation

Theorem 3 provides a technique to interpret ruled surface approximation problems as curve approximation problems in the image space $\mathbb{R}^4$. In more detail we want to study the following. Given m lines L_i and corresponding parameter values τ_i, we will construct a ruled surface surface Φ with generators $X(t)$, such that $X(\tau_i)$ is close to L_i, in the sense of the distance function defined by (23).

Note that the distance $d(A, B)$ of two lines is not invariant under motions in $\mathbb{R}^3$, but essentially depends on the choice of the z-axis and the planes π_0, π_1. We can overcome this disadvantage by the following construction.

We have to determine a unit vector z as third coordinate axis, such that the angles formed by z and given lines L_i are as small as possible. To achieve this, the vector z is computed as solution of a regression problem to the given direction vectors (l_{i1}, l_{i2}, l_{i3}) of lines L_i. If some angle $\angle(L_i, z)$ is larger than a user defined bound $\gamma_0 < \pi/2$ one has to perform a segmentation of the data. Additionally, planes π_0, π_1 have to be chosen in such a way that they bound the domain of interest.

In order to determine $X(t)$ we minimize

$$F := \sum_{i=1}^{m} d(L_i, X(\tau_i))^2 \tag{25}$$

for an unknown ruled surface with generating lines $X(t)$. With respect to the mapping (21) we will restrict the class of ruled surfaces to those whose intersection curves with the planes π_0, π_1 are B-spline curves. Thus, the image curve of $X(t)$ in $\mathbb{R}^4$ is a B-spline curve

$$Y(t) = \sigma(X(t)) = \sum_{i=0}^{n} N_i^n(t) C_i$$

with control points C_i and B-splines of degree n as basis functions over a chosen knot vector.

Let the intersection curves of $X(t)$ with π_0, π_1 be denoted by

$$a(t) = (x_1(t), x_2(t), 0) \quad \text{and} \quad b(t) = (x_3(t), x_4(t), 1). \tag{26}$$

Since $x_i(t)$ are piecewise polynomials, F is a quadratic function in the unknown coefficients of the control points C_i. This implies that the minimization is done by solving a linear system of equations.

The ruled surface strip Φ determined by (26) possesses the following parameterization as point set in $\mathbb{R}^3$,

$$y(t, u) = (1 - u)a(t) + ub(t), \tag{27}$$

where $u \in [0, 1]$ parametrizes the line segments on the generating lines $X(t)$. The shape of $y(t, u)$ essentially depends on the chosen knot sequence and on

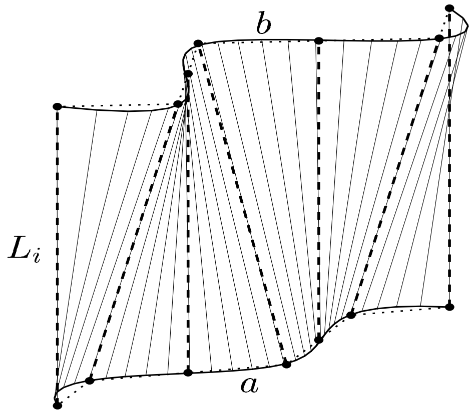

Fig. 7. Ruled surface approximating the given (dashed) lines L_i; the image $\sigma(X(t)) \subset \mathbb{R}^4$ is a cubic B-spline curve consisting of two segments

the parameter values τ_i. To improve the behaviour of the approximant Φ, we can combine (25) with the minimization of a fairness functional. A good choice, although parameter dependent, is usually the functional

$$G(y) = \int_{t_0}^{t_1} \int_0^1 (y_{tt}^2 + 2y_{tu}^2 + y_{uu}^2) du dt.$$

Inserting (27) and elaborating this we obtain a functional involving coordinate functions of $\sigma(X(t)) = Y(t) = (a, b)(t)$,

$$G(Y) = \frac{1}{3} \int_{t_0}^{t_1} (\ddot{a}^2 + \ddot{a}\ddot{b} + \ddot{b}^2) dt + 2 \int_{t_0}^{t_1} (\dot{b}^2 - 2\dot{a}\dot{b} + \dot{a}^2) du.$$

Substituting $Y(t) = (a(t), b(t))$ we obtain

$$G(Y) = \frac{1}{3} \int_{t_0}^{t_1} \langle \ddot{Y}, \ddot{Y} \rangle dt + 2 \int_{t_0}^{t_1} \dot{Y} \cdot M \cdot \dot{Y} dt,$$

where M is the positive semidefinite matrix

$$M = \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \text{ with } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As an improvement of minimizing (25) we can minimize the quadratic function

$$F(X) + \lambda G(X).$$

This amounts again to the solution of a linear system in the unknown coordinates of the control points C_i.

Another possibility to improve the approximation is to apply a correction of the parameter values as mentioned in previous sections and described in [11].

6.3 Approximating line congruences

A two-parameter set of lines or more precisely, the preimage of a two-dim. surface in the Klein quadric, is called a *line congruence*. Its image under mapping σ (21) is a two-dimensional surface in $\mathbb{R}^4$. Thus, approximation problems for line congruences are transferred into surface approximation problems in Euclidean 4-space.

Approximating line congruences have applications in tool motion planning for 5-axis NC machining [31]. There, we have an additional requirement on the congruences: they shall form a fibration in some neighborhood of the surface to be machined. The inclusion of this property into the approximation algorithm is one of our current research topics.

7 Summary: A concept for approximation in spaces of geometric objects

We have treated spaces of different geometric objects: planes, lines and spheres. At first glance, the methods for approximation seem to be quite different. However, in all cases there are some basic steps involved:

- First, construct an appropriate point model for the k-dimensional space S^k of objects under consideration. If this is not yet a projective space, define – at least locally – a mapping into projective k-space P^k.
- By removing a hyperplane from the projective space P^k (removing an appropriate subset of the considered space of objects), we get an affine space A^k.
- Define an appropriate (Euclidean) metric in A^k, which is motivated by a deviation measure between two objects in S^k.
- The computation of m-parameter families of objects in S^k, which approximate some given objects from S^k, is thus transferred into an approximation problem for m-dimensional surfaces in Euclidean k–space.

Note that we presented the cases with increasing level of complication: For spheres, we immediately arrived at an affine space. For planes, we first got a projective space P^3. For lines, we first had a quadric as point model which could then be mapped onto an affine space.

The general concept still remains on a very high, not sufficiently detailed level. However, it is a nice feature that the discussed special cases fit into the following specific general framework. It concerns spaces of *affinely equivalent geometric objects*. This means, that any two objects of the considered class may be mapped into each other by an affine mapping (which needs not be unique, as we will see later).

We outline the concept for objects in $\mathbb{R}^3$, since its generalization to arbitrary dimensions is straightforward. Consider an object Γ in $\mathbb{R}^3$. We think of

Γ as a curve, surface or solid. With respect to some tetrahedron $V_0, \dots, V_3$, it may have the parametric barycentric representation

$$X(u) = \sum_{i=0}^{3} f_i(u) V_i, \quad \text{with} \quad \sum_{i=0}^{3} f_i(u) = 1. \tag{28}$$

The parameters u are from a domain in $\mathbb{R}^l$ with $l = 1$ for a curve Γ, $l = 2$ for a surface Γ and $l = 3$ for a solid Γ. The space $S = S^k$ of geometric objects is generated by the affine images of Γ. Let the tetrahedron $V_0^1, \dots, V_3^1$ be the affine image of the tetrahedron $V_0, \dots, V_3$. Then, the corresponding element Γ^1 of S has the representation

$$\Gamma^1 : \quad X^1(u) = \sum_{i=0}^{3} f_i(u) V_i^1.$$

A simple way to get a point model for S is to interpret the 12 coordinates of the four points $V_0^1, \dots, V_3^1$ as a point $G^1 = \sigma(\Gamma^1)$ in 12-dimensional affine space A^{12}. It is a point model for the affine maps in A^3 (see [29] for a more general point model of projective maps, which contains the present one as a subspace).

The mapping $\sigma : S \to A^{12}$ maps elements of S to points in A^{12}. We still have to introduce a metric in A^{12} based on the deviation of elements in S. This is done as follows. Two elements Γ^1, Γ^2 of S, written as

$$\Gamma^j : \quad X^j(u) = \sum_{i=0}^{3} f_i(u) V_i^j, \quad j = 1, 2,$$

may be mapped onto each other by the affine map α which maps the corresponding tetrahedra onto each other, i.e., $\alpha(V_i^1) = V_i^2$. The vectors connecting corresponding points are

$$D(u) = X^2(u) - X^1(u) = \sum_{i=0}^{3} f_i(u)(V_i^2 - V_i^1). \tag{29}$$

We view $\|D^2(u)\|$ as a function defined on the fundamental object Γ and integrate it over Γ to get a *distance* d between Γ^1 and Γ^2,

$$d^2(\Gamma^1, \Gamma^2) := \frac{1}{N} \int_\Gamma D^2(u) d\Gamma. \tag{30}$$

Here, we have normalized with the integral over Γ,

$$N = \int_\Gamma d\Gamma.$$

Inserting (29) into (30), we realize d^2 as a positive definite quadratic form in the coordinates of the difference vectors $D_i = V_i^2 - V_i^1$,

$$d(\Gamma^1, \Gamma^2) = \frac{1}{N} \left[D_0^2 \int_\Gamma f_0^2(u) d\Gamma \right. \quad (31)$$
$$\left. + 2D_0 \cdot D_1 \int_\Gamma f_0(u) f_1(u) d\Gamma + \ldots + D_3^2 \int_\Gamma f_3^2(u) d\Gamma \right].$$

Hence, we have introduced a *Euclidean metric* in A^{12} and thus have a way to solve approximation problems in S via approximation in Euclidean 12-space.

Theorem 4. *The above defined mapping σ maps elements from a space S of affinely equivalent geometric objects in $\mathbb{R}^3$ to points in A^{12}. The deviation measure (30) between objects Γ^1, Γ^2 in S introduces a Euclidean metric in the image space A^{12}.*

The occurring integrals are extended over a representative Γ of the space S. If Γ has the dimension of the space it spans (i.e., is either a line, a planar domain or a 3D solid), it does not matter which element of S has been chosen as Γ. This is so, since ratios of volume integrals are invariant under affine maps. Otherwise, this choice has an influence on the definition of the metric.

Let us briefly discuss now, in which way our previous investigations are special cases of the present more general concept.

1. For the deviation measure between planes (based on the same domain Γ of interest), we may see the affine maps between two planes realized by a projection parallel to the z-axis. The space of geometric objects is then the set of images of the domain Γ under projections parallel to the z-axis onto all planes which are not parallel to the z-axis. Clearly, we need no reference tetrahedron for the affine image of the planar figure Γ, and now the space S and the image space A^3 are only 3-dimensional. Otherwise, except for the (unnecessary) normalization, the concept for planes is precisely a special case of the general concept.
2. The deviation measure between lines defines affine maps between two lines with help of the auxiliary planes. In this way we set up the affine mapping via two points and their affine images. Clearly, we do not need more for our concept. Note that the same idea applies to the 6-dimensional space of *line segments* in $\mathbb{R}^3$ (see [3,8]).
3. Finally, for the set of spheres we have set up a special affine mapping between two spheres, namely the central similarity (compatible with the orientation). This is the only case where Γ spans 3-space and where we do not have a volume integral over the spanning space. Clearly, we could not use the whole space of affine images of a sphere, since we did not want to work with ellipsoids but with spheres only. In this way, we arrive at a 4-dimensional point model. The deviation measure is exactly the one which corresponds to the definition in the general concept.

There is still a lot of room for work in the area of approximation in spaces of geometric objects. For example, other important spaces (CSG primitives and other fundamental objects in CAD systems) could be investigated. Even the special cases of planes, lines and spheres need more research, in particular for m-parametric families of objects with $m > 1$.

There are also relations to known approaches which deserve further investigations. In particular, the present ideas are closely related to *kinematic mappings* which are used in motion design (see [30] and the references therein). It has to be investigated whether we can improve the construction of approximating motions based on our approach.

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