Smooth surfaces from bilinear patches: discrete affine minimal surfaces

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Abstract

Motivated by applications in freeform architecture, we study surfaces which are composed of smoothly joined bilinear patches. These surfaces turn out to be discrete versions of negatively curved affine minimal surfaces and share many properties with their classical smooth counterparts. We present computational design approaches and study special cases which should be interesting for the architectural application.

Keywords: Discrete differential geometry, affine minimal surface, asymptotic net, architectural geometry, fabrication-aware design

1. Introduction

Freeform structures constitute one of the major trends in contemporary architecture. While we have plenty of ways to handle the pure digital shape modeling tasks, the realization of a complex shape at the large architectural scale is still a challenge. A significant number of problems to be solved are rooted in Geometric Design: Ideally, architects should use systems which generate only those shapes that can be efficiently built with the chosen material and fabrication technology. Such an approach to digital design may be called *fabrication-aware design*. The present paper attempts to make a contribution in this direction. It also continues along the lines of recent research which revealed a close connection between the design and construction of architectural freeform structures and *discrete differential geometry*. It turned out that practical requirements make certain discrete surface representations, such as meshes

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with planar faces and offset properties, very attractive for architectural applications. In fact, the architectural application led to the formulation of some new concepts in discrete differential geometry [2, 3].

Some architects aim at architectural structures which exhibit smooth freeform skins. However, smooth architectural freeform hulls are a big challenge. Since they have to be composed of panels and the production of the panels needs to be feasible, one has to compose smooth surfaces from simple types of surface patches (panels). If one aims at smoothness, planar panels are not suitable. One can replace planar panels with developable panels, in particular cylindrical and conical ones. This led to the introduction of developable strip models. They may be viewed as semi-discrete structures and provide a link between the smooth and fully discrete setting [17]. However, while being smooth along strips, the arising structures still exhibit kinks between adjacent strips. Paneling freeform surfaces with an algorithm that allows one to control the trade-off between smoothness and construction cost has been addressed by Eigensatz et al. [8]. None of these methods will be able to generate a general smooth freeform surface.

In this paper, we pursue a different direction: Prescribing a simple type of panels, namely *bilinear patches*, we ask for those surfaces which can be generated by smoothly joining these bilinear patches. Bilinear patches are parts of hyperbolic paraboloids and have been widely used in architecture, where they are called *hypar shells*. The actual construction exploits their geometric properties: They contain two families of straight lines and are also translational surfaces. To the best of our knowledge, there is no built structure so far where bilinear patches have been joined in a nontrivial way to obtain a smooth surface.

It is clear that smoothly joined negatively curved patches will only generate models of negatively curved surfaces. The shape limitation is even stronger. We will show here that smooth surfaces from bilinear patches are *discrete affine minimal surfaces with indefinite metric*. They agree with the surfaces of Craizer et al. [6] who did not point to the fact that their quad meshes, when filled by patches of hyperbolic paraboloids, do in fact generate smooth surfaces. We will complement the work of Craizer et al. by a different approach, a study of important special cases and their relations to the classical geometric literature, and in particular by proposing methods for computational design of these surfaces.

1.1. Previous work

Negatively curved smooth surfaces from ruled surface strips, but not bilinear patches, have already been addressed by S. Flöry et al. [9, 10, 11]. This work can be seen as contribution to the computation of semidiscrete asymptotic parameterizations; for a mathematical study of this topic, we refer to work by J. Wallner [26]. A rather surprising result has been achieved by E. Huhnen-Venedey and T. Rörig [12]: They showed that discrete asymptotic parameterizations, namely quad meshes with planar vertex stars (A-nets) can be extended to smooth surfaces via rational bilinear patches (under a certain condition on the way the quad strips in the mesh are twisted). Thus, they showed that rational bilinear patches (general ruled quadrics) can be joined to smooth surfaces. This work has not yet been fully exploited for architectural design, and it is not easy to directly extract our special case of bilinear patches. From a purely theoretical perspective, the work of Rörig and Huhnen-Venedey relates via Lie's famous line-sphere transformation to another result on smooth surfaces from simple patches: A circular mesh (quad mesh all whose quads possess a circumcircle) can be extended to a continuously differentiable surface by appropriately filling the quads with patches of Dupin cyclides [4]. The relation between circular meshes and surfaces consisting of cyclidic patches was further extended to circular arc structures and volume structures for architectural applications [5].

As soon as we will have established the relation to discrete affine minimal surfaces, we will mention the key references on this topic. Due to the close relation of our work to *discrete differential geometry*, we would like to point to the monograph by Bobenko and Suris [2] which provides an excellent account of this rapidly expanding field.

From the perspective of CAGD, our paper discusses smooth patchworks from Bezier patches of degree (1,1), and thus it may be seen as a simple, but so far not yet studied contribution to geometrically continuous patchworks.

1.2. Contributions and overview

The contributions of the present paper are as follows:

1. In section 2, we show that smoothly joined bilinear patches are discrete models of negatively curved affine minimal surfaces. Based only on the smooth joining of bilinear patches, we derive a construction of discrete affine minimal surfaces from discrete translational surfaces. This construction features a correspondence between quads which is similar to the discrete Christoffel duality used for the generation of discrete Euclidean minimal surfaces (see e.g. [2]).
2. Section 3 employs these findings for the design of smooth surfaces from bilinear patches and provides tools for exploring the possible shapes.
3. Hyperbolic paraboloids possess excellent structural properties when their axis is vertical. Thus, in section 4, we study the case where all bilinear patches have a vertical axis. The corresponding surfaces are discrete counterparts to improper affine spheres. Viewing the constant axis direction as isotropic direction in isotropic 3-space, the surfaces possess constant relative curvature (the isotropic counterpart to Gaussian curvature) and they can be kinematically generated as translational surfaces via so-called Clifford translation. This generalizes results by K. Strubecker [25] on smooth surfaces of constant relative curvature to the discrete setting.

2. Joining bilinear patches smoothly

We are interested in designing a quad mesh with regular topology and vertices $\mathbf{f}_{ij}$, $i = 1, \ldots, M, j = 1, \ldots, N$, so that the bilinear patches P_{ij} determined by each quad $\mathbf{f}_{ij}, \mathbf{f}_{i+1,j}, \mathbf{f}_{i+1,j+1}, \mathbf{f}_{i,j+1}$ join smoothly. We will denote the edges

$\mathbf{f}_{ij}\mathbf{f}_{i+1,j}$ of the mesh rows by e^1_{ij} and the column edges $\mathbf{f}_{ij}\mathbf{f}_{i,j+1}$ by e^2_{ij}. Note that a bilinear patch (hyperbolic paraboloid) P_{ij} carries two families of straight lines, which we call the e^1-regulus (containing $e^1_{ij}, e^1_{i,j+1}$) and the e^2-regulus (containing $e^2_{ij}, e^2_{i+1,j}$). The lines in each regulus are parallel to a plane (directing plane) E^1_{ij} and E^2_{ij}, respectively. The direction which is parallel to both directing planes is the *axis direction* $\mathbf{a}_{ij}$ of the paraboloid; it is parallel to the vector $\mathbf{f}_{ij} - \mathbf{f}_{i+1,j} + \mathbf{f}_{i+1,j+1} - \mathbf{f}_{i,j+1}$.

Clearly, smoothness requires that the edges through each vertex are coplanar, i.e., the mesh is a so-called *asymptotic mesh* or *A-net* [2], a discrete counterpart to the network of asymptotic curves on a smooth negatively curved surface. Our considerations will be based on the following elementary fact:

Lemma 1. *Two bilinear patches $P_{i,j}$ and $P_{i+1,j}$ join smoothly along a common edge $e^2_{i+1,j} = \mathbf{f}_{i+1,j}\mathbf{f}_{i+1,j+1}$ if and only if*

1. *the edges at the common vertices $\mathbf{f}_{i+1,j}$ and $\mathbf{f}_{i+1,j+1}$ are co-planar and*
2. *the three edges $e^2_{i,j}, e^2_{i+1,j}, e^2_{i+2,j}$ are parallel to a plane.*

The latter property implies that the reguli determined by the common edge have a common directing plane.

Proof. For the two bilinear patches to join smoothly means that along their common edge their respective tangential planes coincide, which are spanned by the patches' rulings. If we let

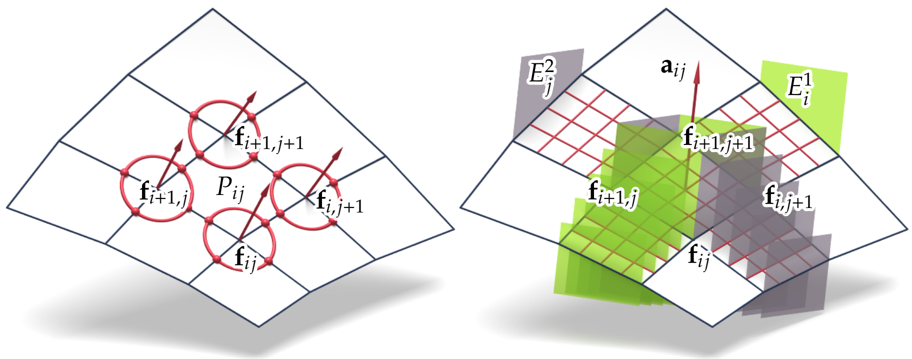
$$f(\lambda, \mu) := \det\left(\lambda e^1_{ij} + \mu e^1_{i,j+1}, e^2_{i+1,j}, \lambda e^1_{i+1,j} + \mu e^1_{i+1,j+1}\right)$$

then clearly the tangential planes at $(1-t)\mathbf{f}_{i+1,j} + t\mathbf{f}_{i+1,j+1}$ coincide if and only if $f(1-t,t) = 0$. Expansion shows $f(\lambda,\mu) = \lambda^2 f(1,0) + \lambda\mu C + \mu^2 f(0,1)$, where $C \in \mathbb{R}$ is independent of λ, μ. $f(1,0) = f(0,1) = 0$ is just the first property (A-net condition). Thus $f(\lambda,\mu) = \lambda\mu C = -\lambda\mu f(-1,1)$, therefore $f(1-t,t) = 0\ \forall t \Leftrightarrow f(-1,1) = 0$, which is the second property. □

Requiring smoothness of patches P_{ij} over the entire mesh, we see that each row of patches $P_{i1}, P_{i2}, \ldots, P_{i,N-1}$ has a common directing plane E^1_i and thus forms a smooth conoidal ruled surface (see Fig. 1). Analogously, each column of patches has a common directing plane E^2_j. Passing to a smooth limit in an appropriate refinement process we will obtain a surface with the following property: Along each asymptotic curve the corresponding 2nd asymptotic directions are parallel to a plane. This is a well known characterization of affine minimal surfaces (see [1], p. 180).

We define a *net polyline* as a set of all vertices $\mathbf{f}_{ij}$ where either i (row polyline) or j (column polyline) is constant, and summarize our first findings:

Theorem 2. *The only quad meshes which can be extended by bilinear patches to overall continuously differentiable surfaces are those A-nets in which the edges that join two neighboring net polylines are parallel to a plane. These surfaces can be seen as discrete affine minimal surfaces with negative curvature.*



(i) planar vertices (A-net) (ii) conoidal strips

Figure 1: Characterization of a quad mesh which can be extended to a smooth surface by filling its faces with bilinear patches: (i) All vertex stars have to be planar. (ii) The edges joining any two neighboring net polygons have to be parallel to a plane. Therefore, on the smoothly extended surface, the strip between two neighboring net polygons contains a continuous family of straight line segments all of which are parallel to that plane and thus the strip defines a conoidal ruled surface.

Our discrete affine minimal surfaces agree with the ones which Craizer et al. [6] derived recently. There, the surfaces are found by discretizing a well-known construction of smooth affine minimal surfaces from translational surfaces arising as co-normal fields. We will now continue the geometric considerations based on smoothly joined patches P_{ij} and arrive at this construction in a purely geometric way.

Let us first consider a single bilinear patch $P := P_{ij}$, and for simplicity call its vertices $\mathbf{v}_1 := \mathbf{f}_{ij}$, $\mathbf{v}_2 := \mathbf{f}_{i,j+1}$, $\mathbf{v}_3 := \mathbf{f}_{i+1,j+1}$ and $\mathbf{v}_4 := \mathbf{f}_{i+1,j,}$. Each vertex has a normal with direction vector

$$\mathbf{n}_i = \lambda(\mathbf{v}_i - \mathbf{v}_{i-1}) \times (\mathbf{v}_{i+1} - \mathbf{v}_i), \tag{1}$$

where indices are taken modulo 4. Vectors $\mathbf{n}_i$ represent the vertices of a *parallelogram* N_P, since opposite edges are parallel,

$$\mathbf{n}_{i+1} - \mathbf{n}_i = \lambda(\mathbf{v}_{i+1} - \mathbf{v}_i) \times (\mathbf{v}_{i+2} - \mathbf{v}_{i-1}) = \mathbf{n}_{i+2} - \mathbf{n}_{i-1}. \tag{2}$$

Moreover, (2) shows that each edge of N_P is orthogonal to a pair of opposite edges of P and thus it is orthogonal to a directing plane of P.

To get more insight into these parallelograms, we take a geometric view (see Fig. 2). The lines $\lambda\mathbf{n}_i, \lambda \in \mathbb{R}$, through the origin form the edges of a pyramid whose face planes are orthogonal to the corresponding edges of the quad P and whose diagonal faces are orthogonal to the non-corresponding diagonals of P (the latter follows from the fact that the tangent planes in the end points of one diagonal of P intersect in the other diagonal). One can transfer these orthogonality relations into an arbitrarily chosen plane Π as follows: We intersect the normal pyramid with Π, resulting in a quad $N^* = \mathbf{n}_1^*, \ldots, \mathbf{n}_4^*$. Also, we project the quad P orthogonally onto Π, resulting in a quad $P^* = \mathbf{v}_1^*, \ldots, \mathbf{v}_4^*$. Then, the resulting two quads have the following property: *corresponding edges are orthogonal and non-corresponding diagonals are orthogonal*. This reminds us of Christoffel dual quads where corresponding edges and non-corresponding diagonals are parallel (see[2], p. 48); we will therefore speak of *Christoffel orthogonal quads* or briefly *C-orthogonal* quads. The definition of C-orthogonality is not restricted to planar quads. Obviously, also the quad P itself is C-orthogonal to N_P.

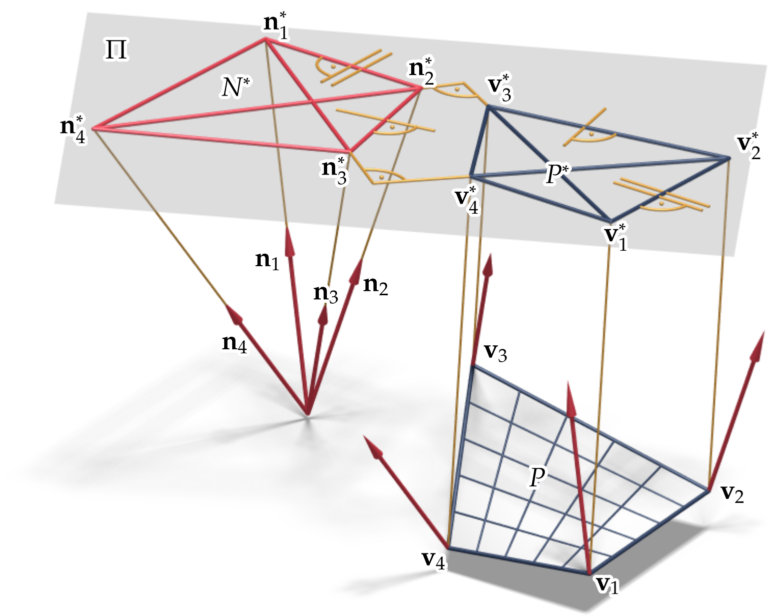


Figure 2: The normals vectors $\mathbf{n}_i$ at the vertices $\mathbf{v}_i$ of a bilinear patch P define a pyramid with vertex at the origin (left). The intersection N^* with a plane Π and the orthogonal projection P^* of P onto Π are Christoffel orthogonal quads: corresponding edges and non-corresponding diagonals are orthogonal.

Let us choose the plane Π orthogonal to the axis of P (i.e., orthogonal to the vector $\mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_2 - \mathbf{v}_4$); this leads to two Christoffel-orthogonal parallelograms P^* and N^*. The edges of the projected quad P^* are parallel to the directing planes of P, those of N^* are orthogonal to the directing planes. Hence, N^* is the parallelogram N_P from above.

If we now take two adjacent quads of an affine minimal net, we can choose the slicing planes (the value λ per bilinear patch) so that the corresponding normal parallelograms N_P share a common edge: This edge is orthogonal to the common directing plane of the two smoothly joined bilinear patches (cf. Lemma 1). Proceeding with this argument we find a *translational net* $\mathbf{n}$ *of normals* associated with the affine minimal net $\mathbf{f}$. For completeness, we mention that the patch dependent scaling factor λ shall be taken as reciprocal value of the affine surface area of the corresponding bilinear patch [6], i.e.,

$$\lambda = 1/\sqrt{\det(\mathbf{f}_2 - \mathbf{f}_1, \mathbf{f}_3 - \mathbf{f}_1, \mathbf{f}_4 - \mathbf{f}_1)}. \tag{3}$$

For our main purposes, namely the design of affine minimal nets $\mathbf{f}$, we do not need these factors, since we take the reverse direction. Starting from a translational net $\mathbf{n}$ with vertices

$$\mathbf{n}_{ij} = \mathbf{n}_i^1 + \mathbf{n}_j^2, \tag{4}$$

we can directly compute the corresponding C-orthogonal net A from the orthogonality relations (also known as discrete Lelieuvre equations),

$$\mathbf{f}_{i+1,j} - \mathbf{f}_{ij} = \mathbf{n}_{ij} \times \mathbf{n}_{i+1,j}, \quad \mathbf{f}_{i,j+1} - \mathbf{f}_{ij} = \mathbf{n}_{i,j+1} \times \mathbf{n}_{ij}. \tag{5}$$

The system of these equations is very easily seen to be integrable: The two ways for obtaining $\mathbf{f}_{i+1,j+1}$ from $\mathbf{f}_{ij}$ (either via $\mathbf{f}_{i+1,j}$ or $\mathbf{f}_{i,j+1}$), are the same,

$$\mathbf{f}_{i+1,j+1} - \mathbf{f}_{i,j} = \mathbf{n}_{i+1,j+1} \times \mathbf{n}_{i+1,j} + \mathbf{n}_{ij} \times \mathbf{n}_{i+1,j} = \mathbf{n}_{i,j+1} \times \mathbf{n}_{i+1,j+1} + \mathbf{n}_{i,j+1} \times \mathbf{n}_{ij},$$

which follows by insertion from (4) (see also [6]). Summarizing, we obtain the following theorem by Craizer et al. [6], to which we have added the interpretation via C-orthogonality and provided a more geometric interpretation based on smoothly joined bilinear patches:

Theorem 3. *Any smooth surface from bilinear patches (discrete affine minimal net* $\mathbf{f}$*) can be constructed from a translational net* $\mathbf{n}$ *(4) that is star-shaped with respect to the origin as a Christoffel orthogonal net with help of the Lelieuvre equations (5).*

The reason why we require a star-shaped translational mesh is illustrated by Fig. 3. We point out that $\mathbf{n}_{i,j}$ (defined through equations (1) and (3)) are Euclidean normal vectors at the vertices $\mathbf{f}_{ij}$ of $\mathbf{f}$, but they are not unit vectors. Sometimes called co-normals or Lelieuvre normals, they define a translational net whose projection onto the unit sphere (centered at the origin) gives the familiar discrete Gaussian image of $\mathbf{f}$ (see Fig. 7, left column).

As discussed above, the relation of C-orthogonality also holds between the orthogonal projection of the mesh $\mathbf{f}$ onto a plane Π and the planar section of the normal image in Π. The latter arises as a central projection of the translational mesh $\mathbf{n}$ from the origin onto Π. We may actually rotatate that section by a right angle in Π and obtain a pair of Christoffel dual meshes:

Corollary 4. *The orthogonal projection of a discrete affine minimal net* $\mathbf{f}$ *onto any plane* Π *yields a mesh whose Christoffel dual mesh may be viewed as perspective image of the translational net* $\mathbf{n}$ *associated with* $\mathbf{f}$ *according to Theorem (3).*

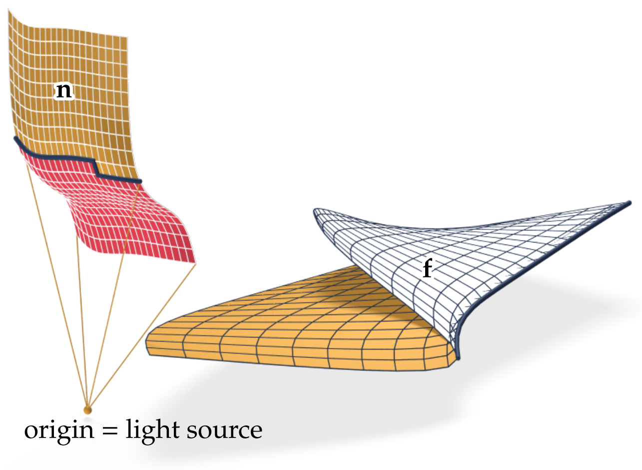


Figure 3: If the translational mesh **n** is not star-shaped with respect to the origin, the corresponding mesh **f** is singular at edges corresponding to **n**'s attached shadow for illumination from the origin.

Remark 5. The bilinear patches (hyperbolic paraboloids) which form the affine minimal net **f** can be seen as discrete Lie quadrics. The fact that the Lie quadrics are all paraboloids is a characterization of affine minimal surfaces (also those which are positively curved and not covered by our approach). The axes of the bilinear patches are discrete affine normals; we have seen that they are also the normals of the faces of the translational net **n**.

Remark 6. It is well known that any A-net can be assigned with forces acting along edges so that the entire force system is in static equilibrium. The reciprocal force diagram is a planar quad mesh; it is a so-called *dual Koenigs net* [2], admitting infinitesimal deformations with rigid faces (R. Sauer [20] called these nets "flächenstarr wackelige Netze"). Our A-nets A have reciprocal PQ nets A^* all whose net polylines are planar: The edges of A^* have the directions $\hat{e}^1_{ij} \parallel e^2_{i+1,j}$ and $\hat{e}^2_{ij} \parallel e^1_{i,j+1}$. Therefore the net polylines with edges $\hat{e}^1_{ij}, i = 1, \ldots, N$, resp. $\hat{e}^2_{ij}, j = 1, \ldots, M$ lie in planes parallel to E^2_j, resp. E^1_i (see Theorem 2 and Figure 1).

Example: Discrete affine and Euclidean minimal surfaces. For practical applications, angles between the two families of rulings should not be too close to 0 or π. Thus, it is natural to ask whether we can get this angle close to $\pi/2$. This would mean that our net **f** is a discrete version of a surface whose asymptotic curves form an orthogonal curve network, i.e., is also a Euclidean minimal surface.

In the smooth setting, this is a well studied subject: Affine minimal surfaces which are also Euclidean minimal surfaces have first been derived by G. Thomsen. H. Schaal [21, 22] presented a simplified derivation and kinematic generations via Clifford translations in a certain Cayley-Klein geometry.

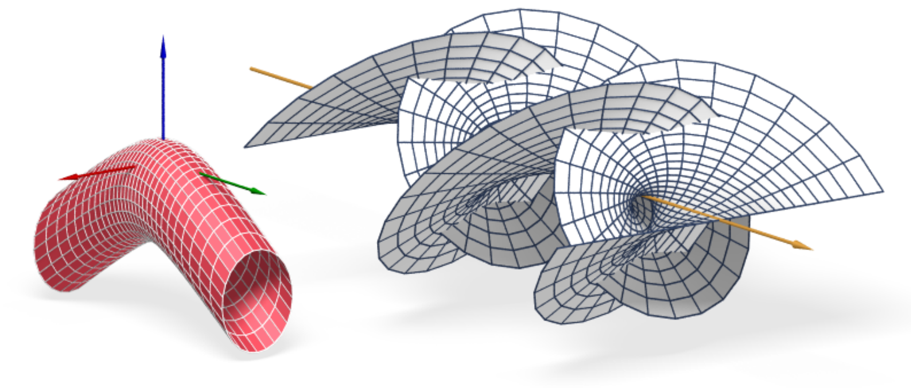


Figure 4: Generation of a smooth patchwork **f** of bilinear patches (right) from a translational net **n** (left). Here, **n** is chosen such that **f** is not only a discrete version of an affine minimal surface, but also of a Euclidean minimal surface. The figure corresponds to a Thomsen surface of Type 1 (**n** taken from equation (6) with $\alpha = 0.6$).

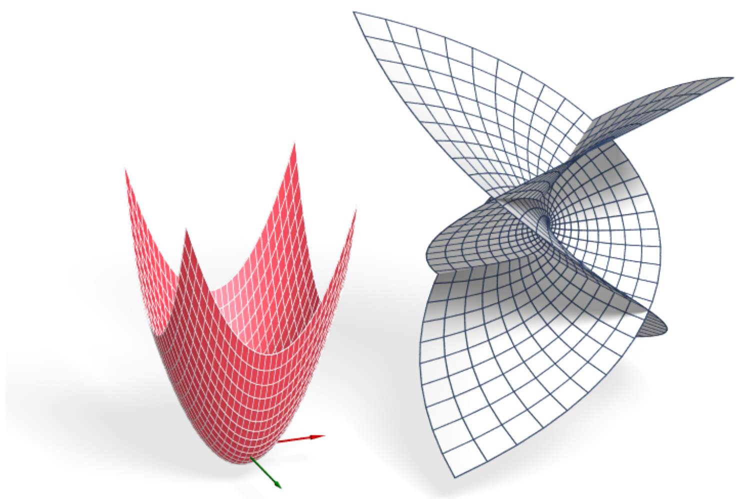


Figure 5: Transforming a translational net **n** (left) which discretizes a rotational paraboloid according to equation (6) yields a smooth union of bilinear patches (right), which can be seen as a discrete version of an Enneper minimal surface.

The simplest way of performing the transfer to the present discrete setting is to use a translational net **n** which discretizes the translational surfaces associated with the smooth counterparts, which come in 2 types. For type 1, the translational surface,

$$\mathbf{n}(u, v) = (\sin u, \sinh v, -(\cos u + \sin\alpha \cosh v)/\cos\alpha), \tag{6}$$

is generated by translating an ellipse along a branch of a hyperbola; α is a constant, $0 < \alpha < \pi/2$. The surface is symmetric with respect to the planes $x = 0$ and $y = 0$, and the ellipse and hyperbola in these symmetry planes possess the origin as focal point. For type 2, $\mathbf{n}$ is a rotational paraboloid with the origin as focal point generated by translating two parabolae along each other,

$$\mathbf{n}(u, v) = (u, v, u^2 + v^2 - \tfrac{1}{4}). \tag{7}$$

Here, the corresponding minimal surface is the well-known Enneper surface.

Discretization of these translational surfaces can be performed by evaluation on an axis-aligned grid in the (u, v)-parameter plane. The translational nets $\mathbf{n}$ and the associated A-nets $\mathbf{f}$ derived from them according to Theorem 3 are shown in Figures 4 and 5, respectively.

3. Surface design and shape exploration

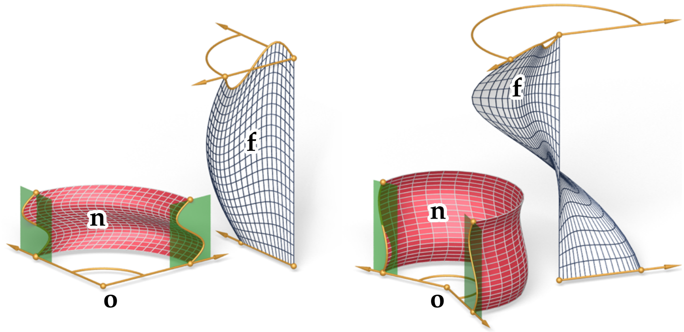


Figure 6: Designing affine minimal nets $\mathbf{f}$ from translational nets $\mathbf{n}$: The approximate angle by which $\mathbf{n}$ is bent around the vertical axis (=angle between the green planes) is roughly the same as the angle between the top and bottom boundary polylines of the corresponding affine minimal mesh $\mathbf{f}$ measured in the horizontal plane, causing twisting of the mesh.

We begin with some tentative observations on how the shape of the translational net $\mathbf{n}$ influences the shape of the affine minimal net $\mathbf{f}$.

If a polyline of $\mathbf{n}$ lies in a plane through the origin $\mathbf{o}$, equation (5) shows that the corresponding polyline on $\mathbf{f}$ is straight; this is the case for two boundary polylines in the two examples of Fig. 6.

If opposite boundary polylines $\mathbf{n}_{11}, \ldots, \mathbf{n}_{1N}$ and $\mathbf{n}_{M1}, \ldots, \mathbf{n}_{MN}$ of the translational net $\mathbf{n}$ are close to parallel lines (which we may imagine as vertical), we can measure an approximate angle between them (around the vertical axis through $\mathbf{o}$; depicted as angle between the green planes in Fig. 6).

For near vertical polylines of $\mathbf{n}$ the corresponding polylines of $\mathbf{f}$ are near horizontal. Measuring an approximate angle of those polylines of $\mathbf{f}$ by using the difference vectors between their end vertices, we find that this angle is roughly the same as the one at the translational net. This rotation around the vertical axis from the bottom boundary to the top boundary polyline causes a twisting of $\mathbf{f}$ (see Fig. 6).

Next let us consider the projection of $\mathbf{n}$ onto the unit sphere, $\bar{\mathbf{n}} = \mathbf{n}/\|\mathbf{n}\|$, that is, the discrete Gaussian image of $\mathbf{f}$. As seen before, if a polyline of $\mathbf{n}$ lies in a plane through $\mathbf{o}$ and thus the vertices of the corresponding polyline of $\bar{\mathbf{n}}$ lie on a great circle, then the corresponding polyline of $\mathbf{f}$ is a straight line (polylines 14, 23, 56 and 58 in Fig. 7).

If in relation to the great circle spanned by the first and last vertex of the polyline, the polyline is bent "outwards", the corresponding polyline of $\mathbf{f}$ will be bent inwards, towards the surface, making its surface area smaller, and vice versa (polyline 67 resp. polylines 12, 34 and 78 in Fig. 7).

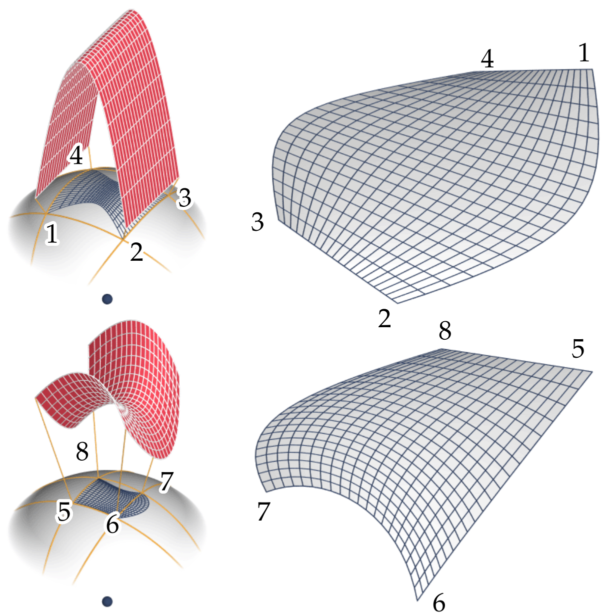


Figure 7: If a polyline of the discrete Gaussian image of $\mathbf{f}$ (=translational mesh $\mathbf{n}$ projected onto the unit sphere) is bent "inwards", the corresponding polyline of the affine minimal mesh is bent "outwards" and vice versa.

Note that the position of the origin $\mathbf{o}$ relative to the translational net $\mathbf{n}$ has an influence on the shape of $\mathbf{f}$. Hence, we should study the change of $\mathbf{f}$ if $\mathbf{o}$ gets changed, or equivalently, $\mathbf{n}$ undergoes a translation. If we let $\mathbf{n}'_{ij} := \mathbf{n}_{ij} + \mathbf{v}$ for some vector $\mathbf{v} \in \mathbb{R}^3$, then the corresponding affine minimal mesh is $\mathbf{f}' = \mathbf{f} + \mathbf{v} \times \mathbf{n}$. $\mathbf{v} \times \mathbf{n}$ is a flat mesh in the plane V with normal direction $\mathbf{v}$. Using the notion of Christoffel orthogonality illustrated in Figure 2, we can see that $\mathbf{v} \times \mathbf{n}$ is Christoffel orthogonal to $\mathbf{n}$ projected onto the same plane V. Disregarding scaling, we can consider $\mathbf{f}'$ to be the mean of $\mathbf{f}$ and $\mathbf{v} \times \mathbf{n}$ as shown in Fig. 8.

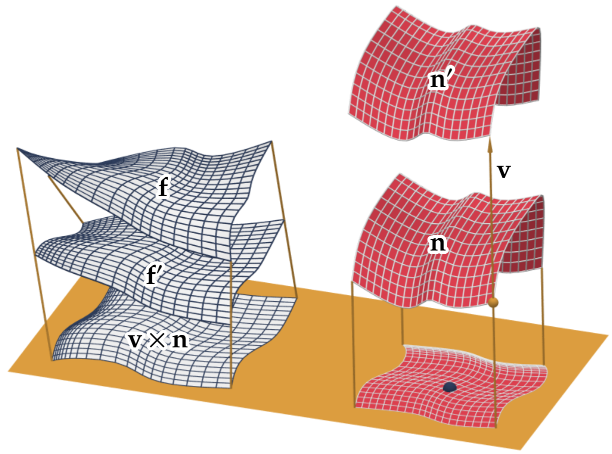


Figure 8: If the normal mesh $\mathbf{n}$ of an affine minimal mesh $\mathbf{f}$ is translated by a vector $\mathbf{v}$, the affine minimal mesh $\mathbf{f}'$ of the resulting normal mesh $\mathbf{n}'$ can be seen as the mean of $\mathbf{f}$ and the planar mesh $\mathbf{v} \times \mathbf{n}$ (which is C-orthogonal to the projection of $\mathbf{n}$ onto the same plane).

3.1. *Affine minimal meshes from patch axes*

The *Björling problem* of classical differential geometry is the task of finding a minimal surface, given a single curve of the surface and the normals along it. The affine version, using affine normals in place of (Euclidean) normals, was studied e.g. by W. Blaschke [1, § 70]. Craizer et al. [7] gave a discrete equivalent of it in the case of a definite Blaschke metric. We will now present the problem in our setting with indefinite metric.

In the continuous case, the given curve can be any curve on the surface, but in the discrete setting the only explicitly available curves are the net polylines (for example $(\mathbf{f}_{i1})_{i=1,\dots,N}$), which in our case are the discrete asymptotic curves, or diagonal polylines, e.g. $(\mathbf{f}_{ii})_{i=1,\dots,N}$. Mesh polylines are not suited for the task as the normal directions are defined by the knot positions alone, so in combination with affine normals the problem is overdetermined. We will therefore use diagonal polylines.

Equations (5) imply

$$\mathbf{f}_{i+1,j+1} - \mathbf{f}_{ij} = \mathbf{n}_{i,j+1} \times \mathbf{n}_{i+1,j+1} + \mathbf{n}_{i,j+1} \times \mathbf{n}_{ij},$$

and therefore,

$$\langle \mathbf{n}_{ij} + \mathbf{n}_{i+1,j+1}, \mathbf{f}_{i+1,j+1} - \mathbf{f}_{ij} \rangle = 0. \tag{8}$$

The discrete affine normals are parallel to the patch axes and defined as

$$\mathbf{a}_{ij} = \det(\mathbf{n}_{ij}, \mathbf{n}_{i,j+1}, \mathbf{n}_{i+1,j})^{-1} \Big(\mathbf{f}_{ij} - \mathbf{f}_{i+1,j} - \mathbf{f}_{i,j+1} + \mathbf{f}_{i+1,j+1} \Big).$$

They satisfy

$$\langle \mathbf{n}_{ij}, \mathbf{a}_{ij} \rangle = \langle \mathbf{n}_{i+1,j}, \mathbf{a}_{ij} \rangle = \langle \mathbf{n}_{i,j+1}, \mathbf{a}_{ij} \rangle = \langle \mathbf{n}_{i+1,j+1,}, \mathbf{a}_{ij} \rangle = 1. \tag{9}$$

This together with (8) gives us three equations for $\mathbf{n}_{i+1,j+1}$,

$$\left(\mathbf{a}_{ij}, \mathbf{a}_{i+1,j+1} - \mathbf{a}_{ij}, \mathbf{f}_{i+1,j+1} - \mathbf{f}_{ij}\right) \cdot \mathbf{n}_{i+1,j+1} = (1, 0, -\langle \mathbf{n}_{ij}, \mathbf{f}_{i+1,j+1} - \mathbf{f}_{ij} \rangle)^T, \tag{10}$$

which have a unique solution provided that

$$\det\left(\mathbf{a}_{ij}, \mathbf{a}_{i+1,j+1} - \mathbf{a}_{ij}, \mathbf{f}_{i+1,j+1} - \mathbf{f}_{ij}\right) \neq 0. \tag{11}$$

This is the same assumption as is made in the continuous case [1, § 70, eq. (55)].

Theorem 7. *Suppose nets* $\mathbf{f}$ *and* $\mathbf{a}$ *are given along the diagonal, that is,* $\mathbf{f}_{ii}$ *and* $\mathbf{a}_{ii}$ *are known. Suppose further that the regularity condition (11) is satisfied.*

Then there is a three-parameter family of affine minimal meshes $\mathbf{f}$*, normal meshes* $\mathbf{n}$ *and face axes* $\mathbf{a}$*. These affine minimal meshes can be constructed by first calculating the normals along the diagonal* $\mathbf{n}_{ii}$ *by solving the systems of linear equations (10), then the off-center diagonals by*

$$\mathbf{n}_{i+1,i} := \tfrac{1}{2}\left(\mathbf{n}_{ii} + \mathbf{n}_{i+1,i+1} - \mathbf{a}_{ii} \times (\mathbf{f}_{i+1,i+1} - \mathbf{f}_{ii})\right), \tag{12}$$

and then all the others by $\mathbf{n}_{ij} + \mathbf{n}_{i+1,j+1} - \mathbf{n}_{i+1,j} - \mathbf{n}_{i,j+1} = 0$*.*

Finally the mesh $\mathbf{f}$ *is constructed from the mesh* $\mathbf{n}$ *by the Lelieuvre equations (5).*

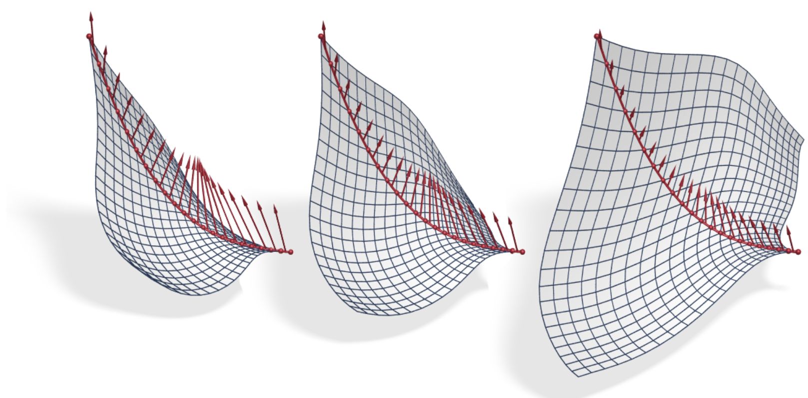


Figure 9: Solutions of the discrete affine Björling problem, with the same diagonal curve but with different affine normal (=patch axis) lengths

Remark 8. The reason for the three free parameters is that at the first and last diagonal knots $\mathbf{f}_{11}$ and $\mathbf{f}_{NN}$ there are in total three of the equations from (10) missing. This can be remedied by specifying one further patch axis at each end ($\mathbf{a}_{00}$ and $\mathbf{a}_{NN}$) and one further diagonal knot ($\mathbf{f}_{00}$ *or* $\mathbf{f}_{N+1,N+1}$).

The construction of an affine minimal net from a diagonal polyline and affine normals along it is illustrated in Fig. 9. It is however mostly interesting from a theoretical perspective. It presents an elementary proof of a result of discrete differential geometry which is a generalization of a theorem in the smooth setting. The latter is described as a beautiful result by Blaschke [1], along with a much less elementary proof.

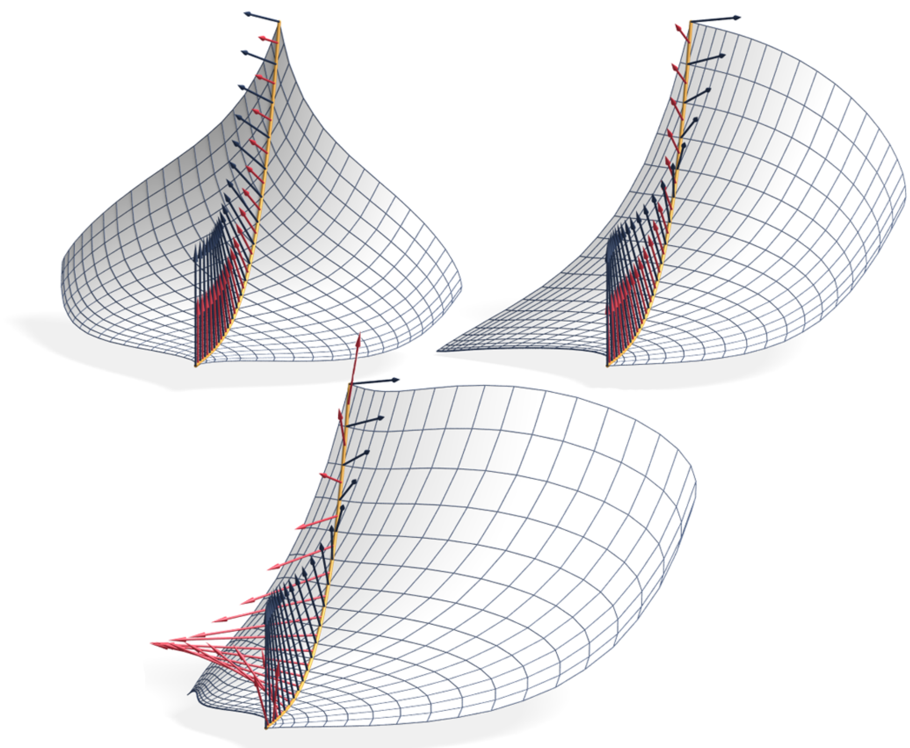


Figure 10: Section 3.2: Affine minimal nets constructed from the same diagonal curve. The upper left net shows the initial values of normals (dark) and patch axes (red). The upper right net is modified from the left one by translating the normals $\mathbf{n}_{ii}$ along $(\mathbf{f}_{i-1,i-1} - \mathbf{f}_{ii}) \times (\mathbf{f}_{i+1,i+1} - \mathbf{f}_{ii})$; the initial positions of the patch axes change subject to the normals. The lower image shows the mesh after modifying the patch axes $\mathbf{a}$ by translating them along $\mathbf{n}_{ii} \times \mathbf{n}_{i+1,i+1}$, keeping the normals fixed.

The Björling problem is not very well suited for practical mesh design, as the relation between the affine normals and the shape of the mesh is not obvious. Many choices of affine normals even result in meshes with singular edges — see Fig. 3 for such a mesh.

The following algorithm is an adaption of the Björling procedure that allows us an easier construction of affine minimal meshes from data given along the diagonal, as it puts more emphasis on controlling the surface normals (see Fig. 10). Assume that $\mathbf{f}_{ii}$ are all known.

- Find initial normal vectors along the diagonal that satisfy equation (8).

- Design the normal directions along the diagonal by translating the $\mathbf{n}_{ii}$ along $(\mathbf{f}_{i-1,i-1} - \mathbf{f}_{ii}) \times (\mathbf{f}_{i+1,i+1} - \mathbf{f}_{ii})$. This keeps (8) valid.

- After that, initial patch axes $\mathbf{a}_{ii}$ are calculated by

$$\mathbf{a}_{ii} := \frac{\mathbf{n}_{ii}\langle \mathbf{n}_{i+1,i+1} - \mathbf{n}_{ii}, \mathbf{n}_{i+1,i+1}\rangle - \mathbf{n}_{i+1,i+1}\langle \mathbf{n}_{i+1,i+1} - \mathbf{n}_{ii}, \mathbf{n}_{ii}\rangle}{\|\mathbf{n}_{ii}\|^2\|\mathbf{n}_{i+1,i+1}\|^2 - \langle \mathbf{n}_{ii}, \mathbf{n}_{i+1,i+1}\rangle^2}.$$

  This makes sure that equation (9) holds, that is $\langle \mathbf{a}_{ii}, \mathbf{n}_{ii}\rangle = \langle \mathbf{a}_{ii}, \mathbf{n}_{i+1,i+1}\rangle = 1$.

- Design the affine normal directions by translating them along $\mathbf{n}_{ii} \times \mathbf{n}_{i+1,i+1}$, keeping the conditions above.

- Construct the rest of the mesh according to the Björling procedure (section 3.1).

3.3. Affine minimal meshes from zigzag polygons

There is another method of constructing affine minimal meshes from diagonal data using a zigzag polygon.

Assume that $\mathbf{f}_{11}, \mathbf{f}_{12}, \mathbf{f}_{22}, \mathbf{f}_{23}, \ldots, \mathbf{f}_{N-1,N}, \mathbf{f}_{NN}$ are known. The normal directions at these knots can easily be calculated, as there are two known edges emanating from each knot, except for the first and last knot. There the missing second edges can be defined for example by specifying one further knot beyond each end, namely $\mathbf{f}_{01}$ and $\mathbf{f}_{N,N+1}$.

A knot $\mathbf{f}_{i+1,i}$ lies in the knot planes of both $\mathbf{f}_{ii}$ and $\mathbf{f}_{i+1,i+1}$. It can be freely chosen anywhere on those planes' intersection line. Once one such knot is fixed, all the others follow uniquely by imposing the properties of an affine minimal mesh, A-net and conoidal strips, see Theorem 2.

Let us summarize these facts:

Theorem 9. *Let $\mathbf{f}_{11}, \mathbf{f}_{12}, \mathbf{f}_{22}, \mathbf{f}_{23}, \ldots, \mathbf{f}_{N-1,N}, \mathbf{f}_{NN}$ be a polygon in $\mathbb{R}^3$ together with normal directions at the first and last vertex $\mathbf{n}_{11} \perp \mathbf{f}_{12} - \mathbf{f}_{11}$ and $\mathbf{n}_{NN} \perp \mathbf{f}_{N-1,N} - \mathbf{f}_{NN}$. Then there is a one-parameter family of affine minimal meshes $\mathbf{f}$ that extends the given polygon and has the given normal directions at the corners $(1,1)$ and (N,N).*

Remark 10. A suitable zigzag polygon can be constructed by a continuous curve $c\colon \mathbb{R} \to \mathbb{R}^3$ and a fixed vector $\mathbf{v} \in \mathbb{R}^3$ as

$$\mathbf{f}_{ii} := c(i) + \mathbf{v} \quad \text{and} \quad \mathbf{f}_{i,i+1} := c(i + \tfrac{1}{2}) - \mathbf{v}.$$

It is also important to note that the torsions of column and row polygons have different signs. The given zigzag polygon has to reflect that fact. For example, if the curve c is planar (as in Fig. 11) this means it cannot have inflection points.

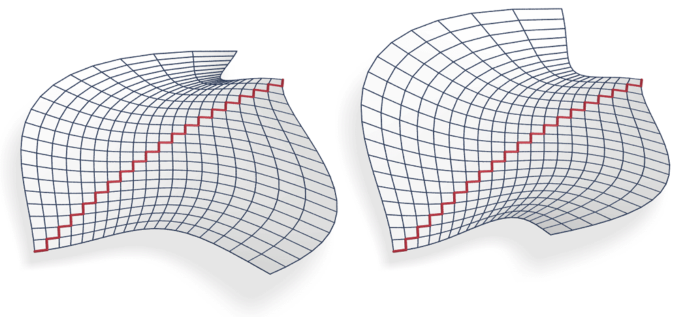


Figure 11: The construction of Theorem (9): affine minimal nets constructed from the same zig-zag diagonal curve and end normals, but with different choices for the free parameter.

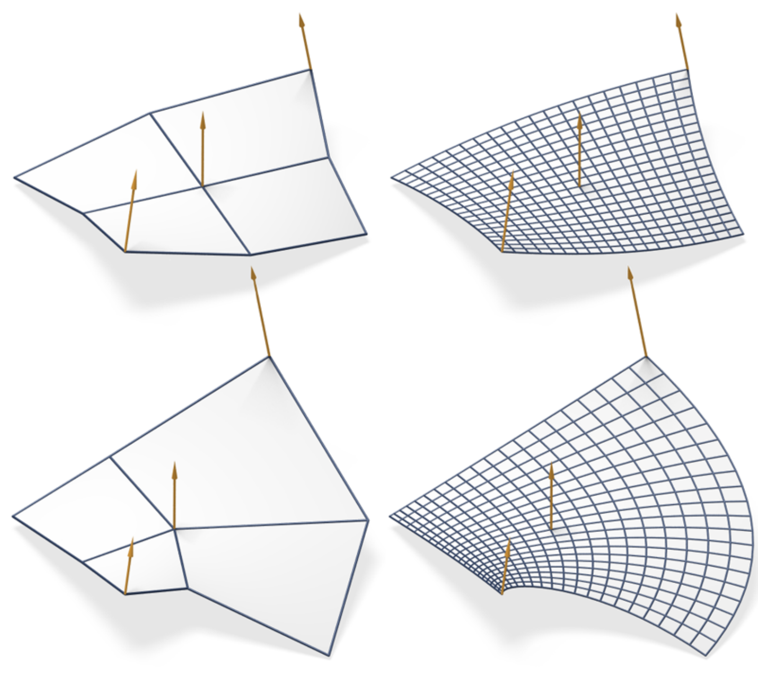


Figure 12: 3-by-3 affine minimal meshes (left) – described by the four corner positions and the normal directions along a diagonal – are refined (right) via refinement of the associated translational net (not shown).

3.4. Affine minimal meshes by refinement

Architectural designs often do not exhibit shape variations on a fine scale. Hence it is feasible and good practice to design the shape on a coarser scale and then compute the final shape via refinement, of course ensuring that the constraints are also met after refinement. This approach has been successfully demonstrated for planar quad meshes (see e.g. [13, 16]) and can also be adapted to the current setting. We just present a simple version in order to demonstrate the basic principle.

An affine minimal mesh with 3-by-3 vertices has 18 degrees of freedom, which can be specified in a variety of ways. We can for example prescribe the positions of the four corners ($\mathbf{f}_{11}, \mathbf{f}_{31}, \mathbf{f}_{13}\mathbf{f}_{33}$) and the directions of the normals along a diagonal (the directions of $\mathbf{n}_{11}$, $\mathbf{n}_{22}$ and $\mathbf{n}_{33}$).

This very simple affine minimal mesh can now be refined by refining the associated translational net $\mathbf{n}$, which is an easy task: One simply refines its generating net polylines $\mathbf{n}_{i1}$ and $\mathbf{n}_{1j}$, by any curve refinement method (e.g. interpolatory subdivision).

The resulting refined affine minimal surface $\mathbf{f}^r$ will satisfy the given normal directions but not the corner positions. If those are of higher importance, the mesh can be affinely transformed to match the corners (as in Fig. 12), but this will cause slight deviations in the normal directions.

Remark 11. In essence, affine minimal nets are special types of constrained meshes. Therefore, designing, editing and exploring the shapes of affine minimal nets can also be based on the constrained mesh exploration framework of Yang et al. [28].

4. Smoothly joined bilinear patches with parallel axes

Let us now study those affine-minimal nets all whose bilinar patches possess the same axis direction. We imagine this direction to be vertical, parallel to the z-axis of a Cartesian system. This special case is motivated by a practical perspective because of the superior statics properties of hyperboloic paraboloids with vertical axis. We aim here at some simple insights for the design of these surfaces and at an understanding of their quite interesting geometry. The structural analysis of the overall smooth surface is left for future work.

Let us start with some remarks on prior work and the theory. Since the paraboloid axes are discrete affine normals, our surfaces are *discrete improper affine spheres* and have been studied as such by Matsuura and Urakawa [14]. However, these authors as well as Craizer et al. [6] did not pursue our approach, namely looking at these surfaces from the perspective of *isotropic geometry*. Defining the z-direction as isotropic direction in an isotropic space, the surfaces appear as isotropic counterparts to the discrete surfaces of constant Gaussian curvature of Sauer [19] and Wunderlich [27]. They are discrete versions of smooth surfaces studied already by Darboux and later by Strubecker [25], who first realized the advantages of using isotropic geometry. We will refrain from a detailed study, but we want to sketch the main results and demonstrate how easily they follow from our geometric framework.

4.1. Construction from planar translational nets

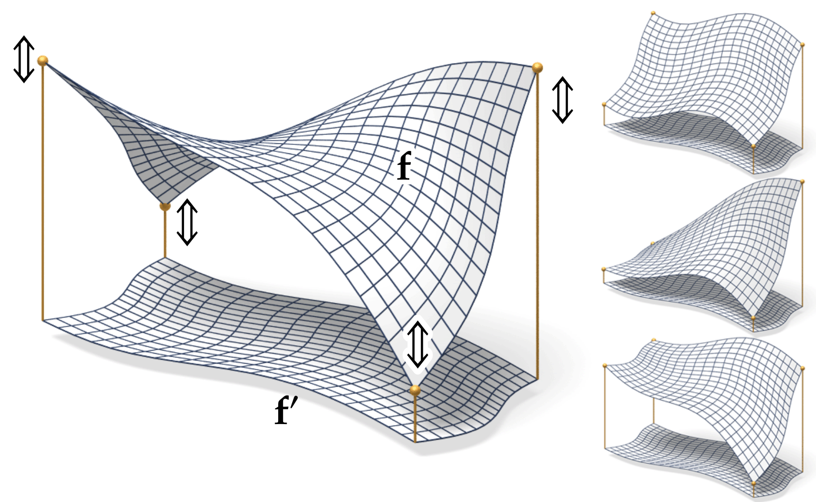


Figure 13: An affine minimal net with vertical axes of the bilinear patches is determined by its top view $\mathbf{f}'$, a translational mesh, and the heights of the four corners.

According to Theorem 3, a smooth surface Φ which is composed of bilinear patches can be obtained from a translational net $\mathbf{n}$ via a rather simple transformation (C-orthogonality). It turned out that the normals of $\mathbf{n}$'s faces are parallel to the axes of the bilinear patches which form the surface Φ. As we now require all axes to be parallel, the faces of $\mathbf{n}$ need to have parallel normals and thus the entire mesh $\mathbf{n}$ has to lie in a plane Π. Conversely, if we apply our construction onto a mesh $\mathbf{n}$ in a plane, say $\Pi\colon z = 1$, the resulting A-net $\mathbf{f}$ has patches whose axes are orthogonal to Π. If one projects a bilinear patch parallel to its axis into a plane, one obtains a parallelogram. Hence, projecting the net $\mathbf{f}$ orthogonally into Π, we must get a translational net $\mathbf{f}'$, which we call the top view of $\mathbf{f}$. Note that $\mathbf{f}'$ and $\mathbf{n}$ are C-orthogonal translational nets in Π.

Let us look at the design of the quad meshes $\mathbf{f}$ and the degrees of freedom we have. To define the top view $\mathbf{f}'$, it is sufficient to prescribe two polygons, say $\mathbf{f}'_{11}, \ldots, \mathbf{f}'_{1N}$ and $\mathbf{f}'_{11}, \ldots, \mathbf{f}'_{M1}$. Additionally, we may fully prescribe the bilinear patch $\mathbf{f}_{11}\mathbf{f}_{12}\mathbf{f}_{21}\mathbf{f}_{22}$ at the common point. Then, by enforcing planar nodes, it is obviously straightforward to build up the net $\mathbf{f}$ above the top view $\mathbf{f}'$.

For practical purposes, it is better to prescribe the z-coordinates at the corners $\mathbf{f}_{11}\mathbf{f}_{1N}\mathbf{f}_{M1}\mathbf{f}_{MN}$ of the entire mesh $\mathbf{f}$, as shown in Figure 13. This can easily be done by first constructing the mesh with an arbitrary (non-planar) choice of $\mathbf{f}_{11}\mathbf{f}_{12}\mathbf{f}_{21}\mathbf{f}_{22}$ and then applying an affine transformation which maps the remaining three corners to the desired locations $\mathbf{f}_{1N}\mathbf{f}_{M1}\mathbf{f}_{MN}$.

4.2. Some basics of isotropic geometry

The geometry of the nets $\mathbf{f}$ becomes interesting if we employ *isotropic geometry*, which has been systematically developed by Strubecker [25, 23, 24] in the 1940s (see also the mongraph by Sachs [18]). It is based on the following group G_6 of affine transformations $(x, y, z) \mapsto (x', y', z')$ in $\mathbb{R}^3$,

$$\begin{aligned} x' &= a + x\cos\phi - y\sin\phi, \\ y' &= b + x\sin\phi + y\cos\phi, \\ z' &= c + c_1 x + c_2 y + z, \end{aligned} \tag{13}$$

called *isotropic congruence transformations (i-motions)*. Obviously, motions in isotropic space I^3 appear as Euclidean motions in the top view.

The 5-parameter subgroup of those i-motions which appear as translations in the top view ($\phi = 0$) can be represented as commutative product of two 3-parameter groups of so-called *Clifford translations*,

$$\begin{aligned} x' &= a + x, \\ y' &= b + y, \\ z' &= c + \epsilon b x - \epsilon a y + z, \quad \epsilon = \pm 1; \end{aligned} \tag{14}$$

one speaks of *right translations* for $\epsilon = 1$ and *left translations* for $\epsilon = -1$.

Many metric properties in isotropic 3-space I^3 (invariants under G_6) are Euclidean invariants in the top view. For example, the *i-distance* of two points $\mathbf{x}_j = (x_j, y_j, z_j)$, $j = 1, 2$, is defined as Euclidean distance of their top views $\mathbf{x}'_j$,

$$\|\mathbf{x}_1 - \mathbf{x}_2\|_i := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{15}$$

Two points (x, y, z_j) with the same top view are called *parallel points*; they have *i*-distance zero, but they need not agree. Since the i-metric (15) degenerates along z-parallel lines, these are called *isotropic lines*. *Isotropic angles* between straight lines are measured as Euclidean angles in the top view.

Isotropic geometry enjoys a *metric duality*. It may be realized by the polarity with respect to the isotropic unit sphere $\Sigma\colon 2z = x^2 + y^2$, which maps the point $\mathbf{p} = (p_1, p_2, p_3)$ to the plane

$$P\colon z = p_1 x + p_2 y - p_3. \tag{16}$$

Points $\mathbf{p}$ and $\mathbf{q} = (q_1, q_2, q_3)$ with i-distance d (from (15)) are mapped to planes P and $Q:\ z = q_1 x + q_2 y - q_3$. The i-angle σ of the two planes P, Q, defined as $\sigma^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2$, equals the point distance d.

Curvature theory of surfaces: A surface Φ without isotropic (z-parallel) tangent planes can be written in explicit form, $\Phi\colon z = f(x, y)$. Defining an isotropic Gauss map from Φ to the isotropic unit sphere Σ via parallel tangent planes, one finds that the derivative of this map (isotropic Weingarten map) is described by the Hessian $\nabla^2 f$ of f. Its eigenvalues are called *i-principal curvatures*. The corresponding orthogonal directions $\mathbf{r}_1, \mathbf{r}_2$ are the eigenvectors of $\nabla^2 f$. With the i-principal curvatures κ_1, κ_2 one defines *isotropic curvature* (or *relative curvature*)

$$K = \kappa_1\kappa_2 = \det(\nabla^2 f) = f_{xx}f_{yy} - f_{xy}^2, \tag{17}$$

and *isotropic mean curvature* H,

$$2H = \kappa_1 + \kappa_2 = \mathrm{trace}(\nabla^2 f) = f_{xx} + f_{yy} = \Delta f. \tag{18}$$

4.3. Viewing smoothly joined bilinear patches with parallel axes as surfaces in isotropic space

Let us consider the constant axis direction of the bilinear patches in our special A-nets $\mathbf{f}$ as isotropic direction in isotropic space. Simple properties of these patches in isotropic geometry, along with the fact that they are smoothly joined, will allow us to derive remarkable geometric properties of the nets $\mathbf{f}$. These appear as discrete counterparts to known results in the smooth case.

A bilinear patch $\mathbf{f}_1, \ldots, \mathbf{f}_4$ whose top view is a parallelegram may be called a skew parallelogram (isogram) in isotropic geometry. Opposite edges have the same isotropic length, say s_1 and s_2, respectively. The tangent planes at the end points of opposite edges form the same isotropic angle (due to the isotropic axial symmetry of the patch with respect to the istropic line which joins the centers of the patch digonals). Let σ_i, $i = 1, 2$, be the isotropic angle formed by the tangent planes at the end points of an edge of length s_i. A simple elementary calculation then shows that

$$\sigma_1 : s_1 = -\sigma_2 : s_2 =: \tau. \tag{19}$$

Since neighboring patches share tangent planes, the ratio $\sigma_i : s_i$ is constant along each row or column polygon in the net; it is τ for one family and $-\tau$ for the other. Obviously, the ratio $\sigma_i : s_i$ is a suitable definition of *discrete isotropic torsion* of a polygon, and thus we can state that *the A-nets under consideration possess two families of discrete asymptotic curves with constant isotropic torsion* $\pm\tau$. Clearly the nets are *isotropic Chebyshev nets* and, for example with arguments as given by Wunderlich [27] in the Euclidean case, one finds that the nets are *discrete counterparts of surfaces with constant isotropic relative curvature* $K = -\tau^2 = const$.

To see that the A-nets are also Clifford translational surfaces, we just need to have a look at a single patch P_{ij}. It is a Clifford surface in isotropic geometry and hence, there is a Clifford translation which maps two opposite boundaries onto each other, say $(\mathbf{f}_{ij}, \mathbf{f}_{i,j+1}) \mapsto (\mathbf{f}_{i+1,j}, \mathbf{f}_{i+1,j+1})$. It can be extended to a one-parameter group which maps the entire surface defined by the patch in itself. As an isotropic motion, it keeps isotropic angles and distances. Thus, it also maps the patch $P_{i,j+1}$ which joins $P_{i,j}$ smoothly along the edge $(\mathbf{f}_{i,j+1}, \mathbf{f}_{i+1,j+1})$ in itself. Continuing in this way, we see that the entire polygon $\mathbf{f}_{i,1}, \ldots, \mathbf{f}_{i,N}$ is mapped to the next polygon $\mathbf{f}_{i+1,1}, \ldots, \mathbf{f}_{i+1,N}$. This proves the following result, extending the main result of Strubecker [25] to the discrete case:

Theorem 12. *Affine minimal nets formed by bilinear patches whose axes are parallel possess remarkable geometric properties if the axis direction is seen as isotropic direction in isotropic 3-space: The nets are discrete surfaces of constant relative curvature $K = -\tau^2$ and may be generated as Clifford translational nets whose generating polygons have constant isotropic torsion $\pm\tau$; each of these polygons lies in a linear line complex.*

The latter statement follows easily from the fact that a linear line complex with isotropic axis has the following property: Two points on a straight line of the complex, apart at a distance s, have null planes whose angle σ is $p \cdot s$, p being the parameter of the complex. Obviously, polygons whose edges lie in such complexes are exactly those which possess constant isotropic torsion (for the smooth limit case, see [25]; for the geometry of linear line complexes, see e.g. [15]).

These considerations also reveal the shape limitations of the A-nets **f** under consideration. We see that the net polylines within the same family have very similar shape, as they are related by special affine maps (isotropic Clifford translations).

Conclusion and future work

Motivated by applications in architecture, we have shown that gluing bilinear patches together in a smooth way leads to interesting discrete versions of affine minimal surfaces which share a lot of essential properties with their classical counterparts. We have addressed the design of these patchworks and also contributed new results to discrete affine minimal surfaces.

A natural path for future work is to use slightly more general patches than just bilinear ones for forming smooth surfaces, but keeping in mind the simple contructability of the patches in view of architectural applications. In particular, we plan to address the global optimization and shape design of patchworks from rational bilinear patches (introduced by Huhnen-Venedey and Rörig [12]). Moreover, one should couple fabrication-aware shape design with other important issues such as structural considerations. For example, we had motivated section 4 with structural properties of the individual panels, but did not yet care about the structural properties of the entire surface. This needs further study.

Acknowledgments

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