Geometry and Freeform Architecture

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Abstract. During the last decade, the geometric aspects of freeform architecture have defined a field of applications which is systematically explored and which conversely serves as inspiration for new mathematical research. This paper discusses topics relevant to the realization of freeform skins by various means (flat and curved panels, straight and curved members, masonry, etc.) and illuminates the interrelations of those questions with theory, in particular discrete differential geometry and discrete conformal geometry.

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1. Introduction

A substantial part of mathematics is inspired by problems which originate outside the field. In this paper we deal with outside inspiration from a rather unlikely source, namely *architecture*. We are not interested in the more obvious ways mathematics is employed in today's ambitious freeform architecture (see Figure 1) which include finite element analysis and tools for computer-aided design. Rather, our topic is the unexpected interplay of geometry with the spatial decomposition of freeform architecture into beams, panels, bricks and other physical and virtual building blocks. As it turns out, the mathematical questions which arise in this context proved very attractive, and the mundane objects of building construction apparently are connected to several well-developed mathematical theories, in particular discrete differential geometry.

The design dilemma. Architecture as a field of applications has some aspects different from most of applied mathematics. Usually having a unique solution to a problem is considered a satisfactory result. This is not the case here, because architectural design is *art*, and something as deterministic as a unique mathematical solution of a problem eliminates design freedom from the creative process. We are going to illustrate this dilemma by means of a recent project on the Eiffel tower.

The interplay of disciplines. We demonstrate the interaction between mathematics and applications at hand of questions which occur in practice and their answers. We demonstrate how a question Q, phrased in terms of engineering and architecture, is transformed into a specific mathematical question Q^* which has an answer A^* in mathematical terms. This information is translated back into an answer A to the original question. Simplified examples of this procedure are the following:

Figure 1. *Freeform architecture.* The Yas Marina Hotel in Abu Dhabi illustrates the decomposition of a smooth skin into straight elements which are arranged in the manner of a torsion-free support structure. The practical implication of this geometric term is easy manufacturing of nodes (image courtesy Waagner-Biro Stahlbau).

Q_1: Can we realize a given freeform skin as a steel-glass construction with straight beams and flat quadrilateral panels?
Q_1^*: Can a given surface Φ be approximated by a discrete-conjugate surface?
A_1^*: Yes, but edges have to follow a conjugate curve network of Φ.
A_1: Yes, but the beams (up to their spacing) are determined by the given skin.

Q_2: For a steel-glass construction with triangular panels, can we move the nodes within the given reference surface, such that angles become $\approx 60°$?
Q_2^*: Is there a conformal triangulation of a surface Φ which is combinatorially equivalent to a given triangulation (V, E, F)?
A_2^*: Yes if the combinatorial conformal class of (V, E, F) matches the geometric conformal class of Φ.
A_2: Yes if the surface does not have topological features like holes or handles.

Overview of the paper. We start in §2 with freeform skins with straight members and flat panels, leading to the discrete differential geometry of polyhedral surfaces. §3 deals with curved elements, §4 with circle patterns and conformal mappings, §5 with the statics of masonry shells, and finally §6 discusses computational tools.

2. Freeform skins with flat panels and straight beams

Freeform skins realized as steel-glass constructions are usually made with straight members and flat panels because of the high cost of curved elements. Often, the flat panels form a watertight skin. Since three points in space always lie in a common plane, but four generic points do not, it is obviously much easier to use triangular panels instead of quadrilaterals. Despite this difficulty, the past decade has seen much research in the geometry of freeform skins based on *quadrilateral* panels. This is because they have distinct advantages over triangular ones – fewer members per node, fewer members per unit of surface area, fewer parts and lighter construction (see Figure 2).

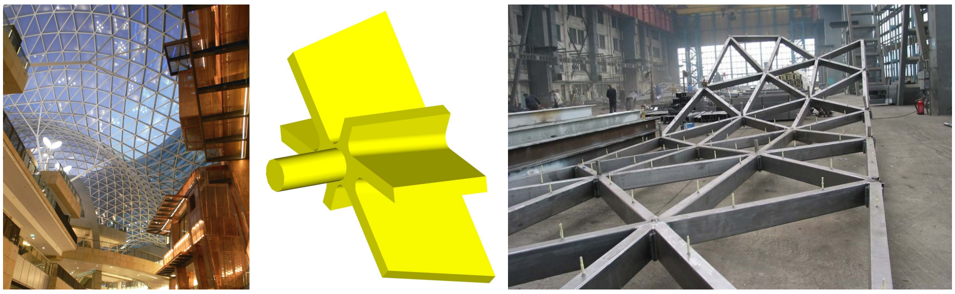


Figure 2. Steel-glass constructions following a triangle mesh can easily model the desired shape of a freeform skin, at the cost of high complexity in the nodes. The *Złote Tarasy* roof in Warszaw (left) is welded from straight pieces and spider-like node connectors which have been plasma-cut from a thick plate (images courtesy Waagner-Biro Stahlbau).

2.1. Meshes. We introduce a bit of terminology: A *triangle mesh* is a union of triangles which form a surface, and we imagine that the edges of triangles guide the members of a steel-glass construction. The triangular faces serve as glass panels. Similarly, *quad* meshes are defined, as well as general meshes without any restrictions on the valence of faces. We use the term *planar quad mesh* to emphasize that panels are flat. Dropping the requirement of planarity of faces leads to general meshes whose edges are still straight. We use V for the set of vertices, E for the edges, and F for the faces. The exact definition of "mesh" follows below.

Meshes from the mathematical viewpoint. While a triangle mesh is simply a 2D simplicial complex of manifold topology, a general mesh is defined as follows. This definition is engineered to allow certain degeneracies, e.g. coinciding vertices.

Definition 2.1. *A mesh in $\mathbb{R}^d$ consists of a two-dimensional polygonal complex (V, E, F) with vertex set V, edge set E, and face set F homeomorphic to a surface with boundary. In addition, each vertex $i \in V$ is assigned a position $v_i \in \mathbb{R}^d$ and each edge $ij \in E$ is assigned a straight line e_{ij} such that $v_i, v_j \in e_{ij}$.*

We say the mesh is a polyhedral surface if it has planar faces, i.e., for each face there is a plane which contains all vertices v_i incident with that face.

2.2. Support structures. An important concept are the so-called torsion-free support structures associated with meshes [30]. Figure 3 shows an example, namely an arrangement of flat quadrilateral panels along the edges of a quad mesh (V, E, F) (which does not have planar faces), such that whenever four edges meet in a vertex, the four auxiliary quads meet in a straight line. We define:

Definition 2.2. *A torsion-free support structure associated with a mesh (V, E, F) consists of assignments of a straight line ℓ_i to each vertex and a plane π_{ij} to each edge, such that $\ell_i \ni v_i$ for all vertices $i \in V$, and $\pi_{ij} \supset \ell_i, \ell_j, e_{ij}$ for all edges $ij \in E$.*

A support structure provides actual support in terms of statics (whence the name), but also has other functions like *shading* [43]. In discrete differential geometry, support structures occur under the name "line congruences".

Figure 3. *Physical torsion-free support structures.* The roof of the Robert and Arlene Kogod Courtyard in the Smithsonian American Art Museum exhibits a mesh with quadrilateral faces and an associated support structure. The faces of the mesh are not planar – only the view from outside reveals that the planar glass panels which function as a roof do not fit together.

Benefits of virtual support structures. Figures 1 and 4 illustrate the Yas Marina Hotel in Abu Dhabi, which carries a support structure in a less physical manner: each steel beam has a plane of central symmetry, and for each node these planes intersect in a common node axis, guaranteeing a clean "torsion-free" intersection of beams. This is much better than the complex intersections illustrated by Figure 2.

Combining flat panels and support structures. It would be very desirable from the engineering viewpoint to work with meshes which have both flat faces and torsion-free support structures. They would be able to guide a watertight steel-glass skin and allow for a "torsion-free" intersection of members in nodes such as demonstrated by Figure 4. The following elementary result however says that in order to achieve this, we must essentially do without triangle meshes.

Lemma 2.3. *Every mesh can be equipped with trivial support structures where all lines ℓ_i and planes π_{ij} pass through a fixed point (possibly at infinity).*

Triangle meshes admit only trivial support structures. More precisely this property is enjoyed by every cluster of generic triangular faces which is iteratively grown from a triangular face by adding neighbouring faces which share an edge.

Proof. For an edge ij, there exists the point $x_{ij} = \ell_i \cap \ell_j$ (possibly at infinity), because ℓ_i, ℓ_j lie in the common plane π_{ij}. If ijk is a face, then $x_{ij} = \ell_i \cap \ell_j = (\pi_{ik} \cap \pi_{jk}) \cap (\pi_{ij} \cap \pi_{jk}) = \pi_{ij} \cap \pi_{ik} \cap \pi_{jk}$ implying that $x_{ij} = x_{ik} = x_{jk} \implies$ all axes incident with the face ijk pass through a common point. For faces sharing an edge that point obviously is the same, which proves the statement. $\square$

Lemma 2.3 has far-reaching consequences since it expresses the incompatibility of two very desirable properties of freeform architectural designs. On the one hand frequently a freeform skin is to be watertight, acting as a roof, which for financial reasons imposes the constraint of planar faces. Unfortunately the planarity constraint is difficult to satisfy unless triangular faces are employed. On the other hand, triangle meshes have disadvantages: We already mentioned the large number

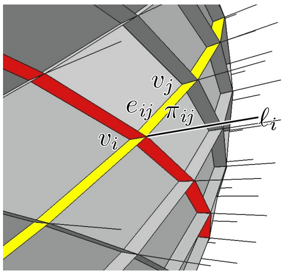


Figure 4. *Torsion-free support structure.* For each edge ij and vertex i of a quadrilateral mesh, we have a plane π_{ij} and a line ℓ_i such that e_{ij}, ℓ_i, ℓ_j are contained in π_{ij} (at left, image courtesy Evolute). This support structure guides members and nodes in the outer hull of the Yas Marina hotel in Abu Dhabi, so that members have a nice intersection in each node (at right, image courtesy Waagner-Biro Stahlbau).

Figure 5. *Not entirely freeform surfaces. Left:* The hippo house in the Berlin zoo is based on a quadrilateral mesh with flat faces, but is not freeform in the strict sense. Its faces are parallelograms, so the mesh is generated by parallel translation of one polyline along another one. Mathematically, vertices $v_{i,j}$ have the form $v_{i,j} = a_i + b_j$. *Right:* the Sage Gateshead building on the river Tyne, UK, is based on a sequence of polylines which are scaled images of each other, similar to a mesh with rotational symmetry.

of members. Lemma 2.3 states that only in special circumstances it is possible to reduce node complexity by aligning beams with the planes of a support structure.

2.3. Quadrilateral meshes with flat faces. Research related to meshes with planar faces is not new, as proved by the 1970 textbook [33] on *difference geometry* by R. Sauer which in particular summarizes earlier work starting in the 1930's. That work was pioneering for discrete differential geometry, which meanwhile is a highly developed area [11]. Relevant to the present survey, questions concerning quad meshes with planar faces ten years ago marked the starting point of a line of research motivated by problems in engineering and architecture [26], which again led to new developments in discrete differential geometry, see §2.4 below.

The meaning of "freeform". Research on quad meshes with flat faces has been rewarding mathematically, but unfortunately hardly any truly freeform meshes of that type have been realized as buildings. Their welcome qualities have nevertheless been used for impressive architecture, but meshes built so far enjoy special

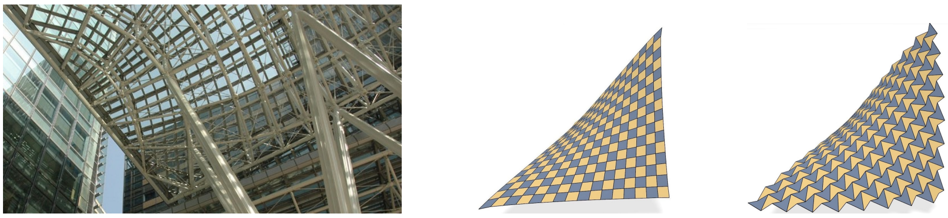


Figure 6. *Quad meshes.* Left: The canopy at “Tokyo Midtown” is based on a quad mesh with planar faces. *Center*: This quad mesh has nonplanar faces, and all meshes nearby which have planar faces are far from smooth. *Right:* This mesh has planar faces and is no direct discrete analogue of a continuous smooth surface parametrization (for such regular patterns see [24]).

geometric properties (like rotational symmetry) which allows us to describe their shape using much less information than would be required in general, see Figure 5.

Smoothness limiting design freedom. A typical situation in the design process of freeform architecture is the following: A certain mesh has been created whose visual appearance fits the intentions of the designer and which eventually is to be realized as a steel-glass construction with flat panels. The designer therefore wants the vertex positions to be altered a little bit so that the faces of the mesh become planar, but its visual appearance does not change. Unfortunately this problem is typically not solvable. This is not because the nonlinear nature of this problem prevents a numerical solution – the reasons are deeper and of a more fundamental, geometric nature: E.g. it is known that a “smooth” mesh of regular quad combinatorics which follows a smooth surface parametrization can have planar faces only if its edges are aligned with a so-called conjugate curve network of the reference surface. The network of principal curvature lines is the major example of that, cf. Figure 8. Since its singularities are shared (in a way) by all conjugate curve networks [45], the principal curvature lines already give a good impression of what a planar quad mesh approximating a given surface must look like. If the designer's mesh is not conjugate, there is no smooth mesh nearby which has planar faces – see Figure 6. There is no easy way out of this dilemma other than reverting to triangular faces, or redesigning the mesh entirely so that its edges follow a conjugate curve network, or forgoing smoothness.

Example: The Cour Visconti in the Louvre. The “flying carpet” roof of the Islamic arts exhibition in the Louvre (Figure 7) provides a good illustration of the choices which have to be made when realizing freeform shapes with flat panels. In this particular case, the architects' design had triangular faces, but from the engineering viewpoint quadrilaterals would have been better (fewer parts, lighter construction, less complex nodes). Changing the visual appearance of the mesh was out of the question, for two reasons: Firstly conjugate curve networks of the flying carpet shape would not have led to meshes with sensible proportions of members, quite apart from their questionable aesthetics. Secondly there are both artistic and legal reasons not to change an architectural design after the decision to realize it has

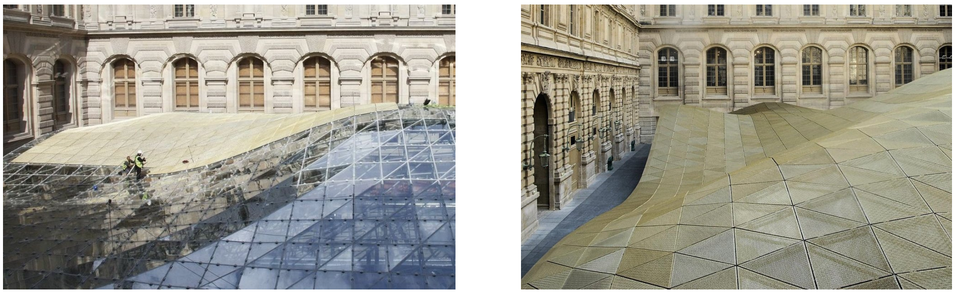


Figure 7. A mesh with planar faces in the Louvre, Paris, by Mario Bellini Architects and Rudy Ricciotti. It has only as many triangular faces as are necessary to realize the architect's intentions (images courtesy Waagner-Biro Stahlbau and Evolute).

been made. In the end, many triangles (as many as it was possible to choose in a periodic way) were merged into flat quadrilaterals. This change is invisible since triangular shading elements were put on top of the glass panels.

Conjugate curve networks. As to the relation between quad meshes and curve networks, consider a mesh with vertices $v_{i,j}$, where i,j are integer indices. Using the forward difference operators $\Delta_1 v_{i,j} = v_{i+1,j} - v_{i,j}$ and $\Delta_2 v_{i,j} = v_{i,j+1} - v_{i,j}$, we express co-planarity of the four vertices $v_{i,j}$, $v_{i+1,j}$, $v_{i+1,j+1}$, $v_{i,j+1}$ by the condition

$$\Delta_1 \Delta_2 v \in \operatorname{span}(\Delta_1 v, \Delta_2 v).$$

One can clearly see the analogy to a parametric surface $x(t,s)$ with the property

$$\partial_1 \partial_2 x \in \operatorname{span}(\partial_1 x, \partial_2 x)$$

which is equivalent to $\mathrm{II}(\partial_1 x, \partial_2 x) = 0$, i.e., the parameter lines are conjugate w.r.t. the second fundamental form (this is the definition of a conjugate network). If in addition, $\mathrm{I}(\partial_1 x, \partial_2 x) = 0$ we get the network of principal curvature lines. In any case, Taylor expansion shows that for sequences $\{t_i\}$, $\{s_j\}$, the quad mesh with vertices $v_{i,j} := x(t_i, s_j)$ has faces which are planar up to a small error which vanishes as the stepsize diminishes. For the more difficult converse statement (convergence of meshes to smooth conjugate parametric surfaces) we refer to [11].

Multilayer constructions. In geometric research on freeform architecture, one focus was the problem of meshes at constant distance from each other [26, 30]. Like the support structures discussed above, such multilayer constructions can be physical or virtual — e.g. a steel-glass construction with members of constant height meeting cleanly in nodes is actually guided not by one mesh, but by two meshes whose edges are at constant distance (Figure 9). This topic has been very fruitful mathematically, and we discuss it in more detail in the next section.

2.4. Discrete differential geometry of polyhedral surfaces. The field of discrete differential geometry aims at the development of discrete equivalents of

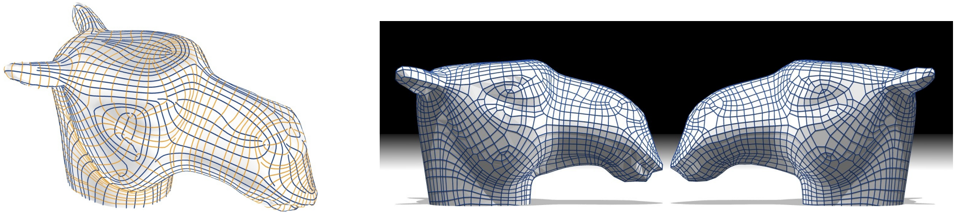


Figure 8. A smooth surface can be approximated by a quad mesh with planar faces, if we first compute the principal curvature lines (at left), then align a mesh with those curves (center) and finally apply numerical optimization (right). One can hardly see the difference between the two meshes, which is why numerical optimization succeeds [26].

notions and methods of smooth surface theory. Discrete surfaces with quadrilateral faces can be treated as discrete analogues of parametrized surfaces. One focus of discretization is classical surface theory which is now associated with the theory of integrable systems. Remarkably this approach led to a better understanding of some fundamental structures [11] and has also impacted on the smooth theory.

Parallel meshes. This concept is the basis of many constructions, from support structures to offsets to curvatures of polyhedral surfaces:

Definition 2.4. *Two meshes M, M' with planar faces and the same combinatorics are parallel, if all corresponding edge lines e_{ij} and e'_{ij} are parallel.*

It is not difficult to see that parallel meshes and support structures are related: We can construct a support structure by connecting corresponding elements in a pair of parallel meshes, and for simply connected meshes also the converse is true.

Offset pairs. If a pair of meshes has approximately constant distance from each other, they are interpreted as an offset pair. If one partner is approximating the unit sphere, they are seen as surface and Gauß image, in the spirit of relative differential geometry. There are several ways to exactify the notion of "approximating the unit sphere" (see also Figure 9):

Definition 2.5. *A mesh M^* whose vertices are contained in the unit sphere is called inscribed. M^* is called circumscribed resp. midscribed if the planes resp. lines associated with faces resp. edges are tangent to the unit sphere. In any case, a parallel pair M, M^* with M^* approximating the unit sphere defines a parallel offset family of meshes $M^{(t)}$, by defining vertex positions $v_i^{(t)} = v_i + tv_i^*$.*

The following is easy to see: In the inscribed (midscribed, circumscribed) case the points (lines, planes) assigned to the vertices (edges, faces) of both M and $M^{(t)}$ have distance t. Such meshes are therefore directly relevant to multilayer constructions in freeform architecture (see Figure 9 and [26, 30, 32]).

Discrete differential geometry of circular meshes, conical meshes, and Koebe polyhedra. There are some nice characterizations of meshes which possess a parallel mesh M^* with special properties. Quad meshes with inscribed M^* are circular,

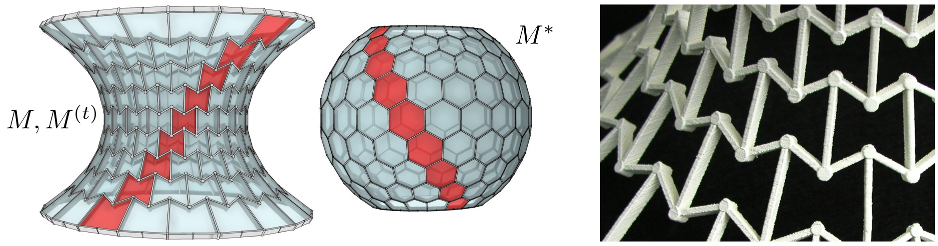


Figure 9. A discrete surface M, its offset $M^{(t)}$ and the Gauß image mesh M^*. Since M^* is midscribed to the unit sphere, corresponding edges of M and $M^{(t)}$ are at constant distance t from each other. The detail at right shows a physical realization of the mesh pair $M, M^{(t)}$ by members of constant height which fit together in the nodes.

i.e., all faces have a circumcircle (the converse is true for simply connected quad meshes). Meshes with circumscribed M^* are conical, i.e., all vertices possess a cone of revolution tangent to the faces there (the converse is true for all meshes). A very interesting case is meshes where M^* is midscribed to the unit sphere. In that case M^* is uniquely determined, up to Möbius transformations, by its combinatorics alone [10]. It is known that the circular and conical quad meshes are discrete versions of principal parametric surfaces (for a convergence statement, see [6]). The Koebe polyhedron case corresponds the so-called Laguerre-isothermic parametrizations which do not exist on all surfaces.

Existence of meshes at constant distance. There are practical implications of this discussion, namely regarding the question if a given shape can be approximated by a quad mesh M with special properties. For circular and conical meshes this is possible in an almost unique way (see Figure 8 for a circular example), but only special surfaces can be approximated by a mesh where M^* is Koebe. This statement immediately translates to the existence of multilayer constructions: For a given shape, meshes at constant vertex-vertex distance or face-face distance exist, with the direction of their edges being essentially unique. Meshes with constant edge-edge distance exist only for special shapes.

A remark on discretization. Throughout its development, discrete differential geometry has studied discrete surfaces defined by properties analogous to properties which define classes of smooth surfaces. The latter may be equivalently defined by various properties, leading to several different discrete versions of it (so that still the resulting discrete surfaces converge to their smooth counterparts upon refinement). “Good” discretizations retain more than one of the original properties.

Integrable systems vs. curvatures. A major focus in discretization is the “integrable” PDEs that govern surface classes. For example, the angle ϕ between asymptotic lines in a surface of constant Gauß curvature obeys the sine-Gordon equation $\partial_t \partial_s \phi = \sin \phi$ after a suitable parameter transform. Consequently, Bobenko and Pinkall [8] based their study of discrete K-surfaces [44] on a discrete version of that equation. Other examples are furnished by *cmc surfaces* (defined by constant mean curvature) and other isothermic surfaces [19]. Apparently integrable

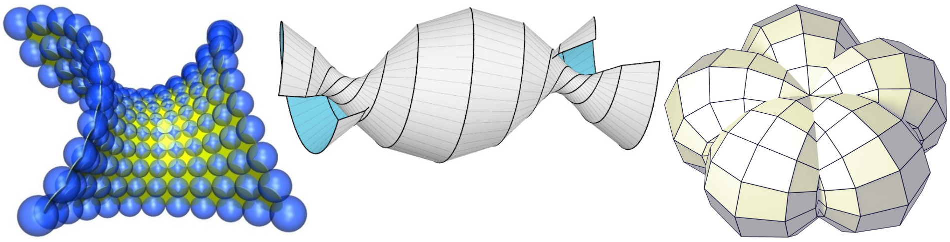


Figure 10. *Discrete surfaces of constant mean curvature H. Left:* An s-minimal surface by T. Hoffmann where Def. 2.6 yields $H = 0$. *Center:* Def. 2.6 does not apply to this cmc surface associated to an unduloid by W. Carl [12], but [22] yields $H = \text{const}$. *Right:* This discrete "Wente torus" by C. Müller [27] has $H = \text{const}$. w.r.t. both theories.

systems provide the richest discrete surface theory. This approach, systematically presented in [11], creates discrete surface classes named after curvatures, whose actual definition however does not involve curvatures at all. It is therefore remarkable that notions of curvature have been discovered which assign the "right" values of curvature to such discrete surfaces after all.

Curvature and the surface area of offsets. In the smooth case, the surface area of an offset surface $M^{(t)}$ at distance t is expressed in terms of mean curvature H and Gauß curvature K, via the so-called Steiner formula. Since a similar relation holds in the discrete case, in [30, 9] we gave a definition of curvatures of polyhedral surfaces which is directly inspired by the classical Steiner formula,

$$\operatorname{area}(M^{(t)}) = \int_M \Big(1 - 2tH + t^2 K\Big) d\operatorname{area}.$$

Definition 2.6. *Assume that a mesh $M = (V, E, F)$ has planar faces, a Gauß image M^*, and an offset family $M^{(t)}$. For a face f with vertices $v_{i_1}, \dots, v_{i_n}$, let*

$$\operatorname{area}(f) = \frac{1}{2}\Big(\det(v_{i_1}, v_{i_2}, N_f) + \cdots + \det(v_{i_n}, v_{i_1}, N_f)\Big) = \langle f, f\rangle,$$

*where N_f is a normal vector common to the face f, the corresponding face f^* in the Gauß image, and the face $f^{(t)} = f + tf^*$ in the offset mesh. We use the notation $\langle\ ,\ \rangle$ to indicate the symmetric bilinear form induced by the quadratic form "*area*" in $\mathbb{R}^{3n}$. We define the mean curvature H_f and the Gauß curvature K_f of the face f by comparing coefficients in the polynomials*

$$\operatorname{area}(f^{(t)}) = \langle f, f\rangle + 2t\langle f, f^*\rangle + t^2\langle f^*, f^*\rangle = \big(1 - 2tH_f + t^2 K_f\big)\operatorname{area}(f)$$
$$\Longrightarrow\ H_f = -\frac{\langle f, f^*\rangle}{\operatorname{area}(f)}, \quad K_f = \frac{\operatorname{area}(f^*)}{\operatorname{area}(f)}.$$

Note that for convex faces, $\langle\,,\rangle$ is the well known *mixed area.* Interesting instances of Definition 2.6 are those where a canonical Gauß image M^* exists. Examples include the minimal surfaces of [7], the remarkable class of minimal surfaces dual to Koebe polyhedra of [5], and the cmc surfaces of [21].

Ongoing work on curvatures. Definition 2.6 is also the basis of further work, including discrete surfaces with hexagonal faces [29, 27]. Recently, Hoffmann et al. [22] developed a concept of curvature which applies to quad meshes (not necessarily planar) equipped with unit normal vectors in vertices. It coincides with Def. 2.6 in the circular mesh case, but also covers the K-surfaces of [44, 8] as well as other significant constructions like associated families of minimal and cmc surfaces [22, 12]. See Figure 10 for illustrations.

3. Freeform skins from curved panels

Curved beams and panels are employed in freeform architecture despite the fact that straight members and flat panels are cheaper and easier to handle. This has to do with artistic reasons and the required quality of the final surface: Discontinuities in the first and second derivative are clearly visible as discontinuities and kinks in reflection lines. Such requirements are known in the automotive industry, but in architecture there is no mass production and the cost of manufacturing curved elements is much higher. It is usually not easy to balance the required surface quality, the complexity of the shape, and the budget.

Mathematical concepts which apply to curved elements are mostly of a differential-geometric nature and often are limit cases of notions known in discrete differential geometry. Optimization, both discrete and continuous, plays an important role. Let us start our discussion by listing a few manufacturing techniques:

— *Concrete* can be poured onto curved formworks. These can e.g. be constructed by approximating the reference shape by a ruled surface. This allows us to use straight elements (Figure 11) or, on a smaller scale, hot wire cutting.

— *Sheet metal* can be bent into the shape of a developable surface, see Figure 12.

— *Double-curved glass panels* are expensive. They are made by hot bending using molds. On large structures, this technique has been employed only in situations were molds can be used to produce more than just one panel, see Figure 13.

— *Single-curved glass* panels can be made by bending in various ways (see Figure 12). In particular, cylindrical glass (see Figure 14) is comparatively cheap

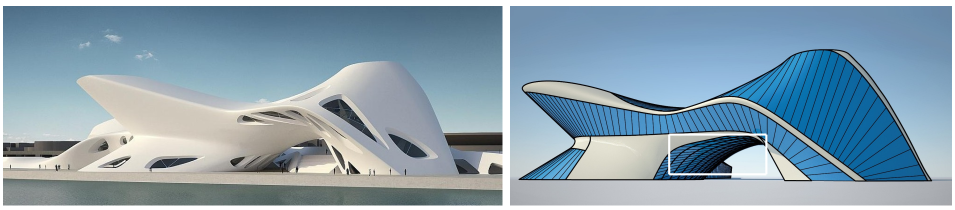


Figure 11. *Formworks for concrete.* Approximating surfaces of nonpositive Gauß curvature with ruled surfaces yields easily-built underconstructions. A union of ruled surface strips can be a C^1 smooth surface even if the rulings exhibit kinks (see marked area). This design by Zaha Hadid was intended for a museum in Cagliari (images courtesy Evolute).

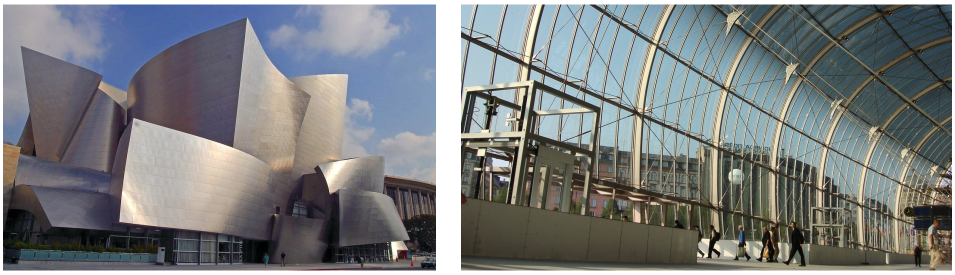


Figure 12. *Single-curved surfaces.* Left: The outer skin of the Disney concert hall in Los Angeles by Frank Gehry is covered by metal sheets and consists of approximately developable surfaces (image courtesy *pdphoto.org*). Right: The glass canopy at the Strasbourg railway station exhibits "cold bent" glass panels (image courtesy RFR).

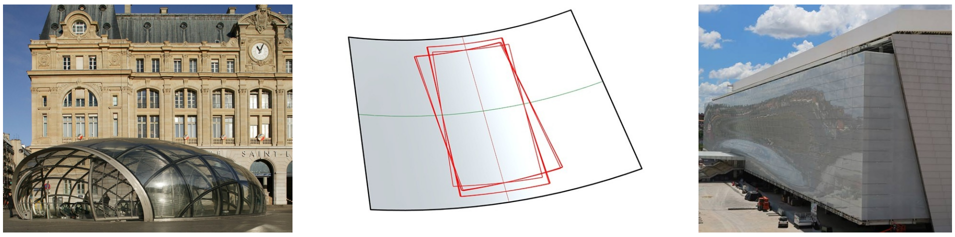


Figure 13. *Double-curved panels. Left:* The entrance to the Paris metro at Saint Lazare station has rotational symmetry, which enables us to manufacture several glass panels with the same mold (image courtesy RFR). *Center:* symbolic image of a mold with panel outlines. *Right:* The 855 panels comprising the facade of the Arena Corinthians in Sao Paolo could be made with 61 molds [34].

to make using machines which for the cooling process use the fact that cylinders permit a two-parameter group of rigid motions moving the surface into itself.

Opimization and manufacturing of panels. The shape of a freeform skin and its dissection into panels is usually very visible and thus is part of an architect's design. Manufacturing panels is an engineering responsibility. Typically, if the intended free form can be achieved by "simple" (e.g., cylindrical) panels, then the individual panels' geometric parameters are found via an optimization problem which usually is conceptually straightforward but might be cumbersome to solve. Its target function involves gaps and kink angles between panels, as well as proximity to the original reference geometry, see Figure 14.

If custom panels are made in molds via hot bending, it is essential to re-use molds, by clustering similar panel shapes and determining a small number of molds capable of manufacturing all panels. The corresponding discrete-continuous optimization problems are hard. They have been studied by Eigensatz et al. [14] and are used for actual buildings, see Figure 13, right and [34].

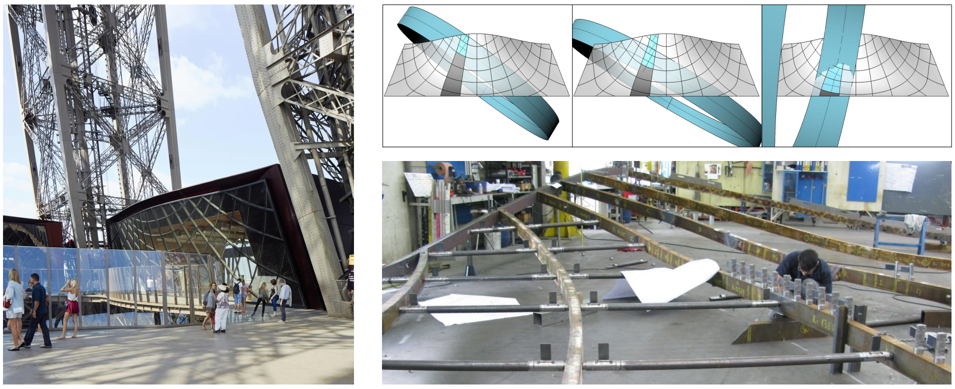


Figure 14. *Panels and beams for the Eiffel tower pavilions.* The facade of the pavilions on the first floor of the Eiffel tower feature cylindrical glass panels and curved beams with rectangular cross-section. *Top Right:* Best approximation of the original reference surface by cylinders ([2], image courtesy Evolute). *Bottom Right:* Since all four surfaces of each beam are manufactured by bending flat pieces of steel plate, they are developable surfaces. They also constitute a semidiscrete support structure (images courtesy RFR).

The Eiffel tower refurbishment and a design dilemma. The first floor of the Eiffel tower exhibits three pavilions, completed in 2015, with a smooth glass facade consisting of cylindrical panels (see above) dissected by curved beams with rectangular cross-section. Figure 14 illustrates the fact that all four surfaces of such a beam are *developable*, since they are manufactured by bending flat sheets. Two of these surfaces are orthogonal to the facade, and it is known from elementary differential geometry that they must then follow principal curvature lines. This implies that the beams, up to spacing, are *uniquely determined* by the facade.

This relation presents a side-condition not easy to satisfy by a designer. In the case of the Eiffel tower pavilions, design freedom was restored by the fact that the principal curvature lines are very sensitive to small changes in the surface, and one could change the facade in imperceptible ways until the beam layout coincided with the original design intentions [36]. It should be mentioned that the Eiffel tower pavilions project benefitted from a cooperation between architects, engineers and mathematicians at an early stage, which possibly was responsible for its success.

Differential geometry of semidiscrete surfaces. Several geometric objects encountered in the discussion above can be seen as semidiscrete surfaces, which depend on one continuous and one discrete parameter. Figure 15 shows what is meant by that, how a semidiscrete surface arises as limit of discrete surfaces, and how it is visualized as a union of ruled strips. Various discrete surface classes have semidiscrete incarnations relevant to architecture:

— Developable ruled surfaces arise as limits of quad meshes with planar faces. They were investigated with regard to architecture (see Figure 15, right, and [26, 31]). They also occur in semidiscrete versions of support structures (cf. our discussion of beams of the Eiffel tower pavilions).

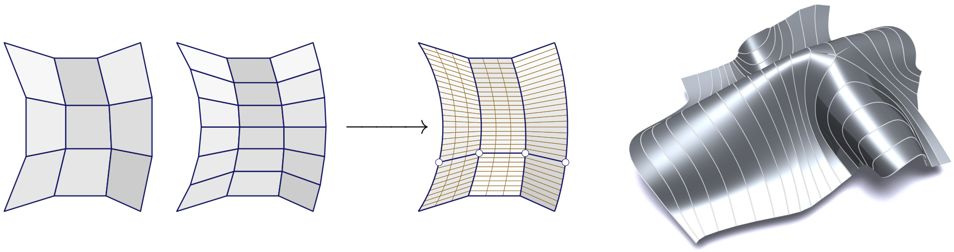


Figure 15. *Left*: Planar quad meshes upon refinement converge to a semidiscrete surface where each strip is developable. *Right*: Approximation by developable strips [31].

— A union of ruled surfaces can be smooth, if it is a semidiscrete version of a mesh with planar vertex stars [15]. An application is shown by Figure 11.
— The geometry of semidiscrete surfaces is an active topic of research (cf. K-surfaces [42], curvatures [25, 28, 12], isothermic surfaces, etc.)

4. Regular Patterns

Regular patterns of geometric objects can mean different things, and there is hardly a limit to creativity: Figure 6 shows a mesh where the regularity lies in the repetitive features of edge polylines [24]. For the mesh consisting of equilateral triangles in Figure 20, regularity means constant edgelength. For other examples see Figure 16. This section deals with a particular kind of patterns related to circle packings. We first discuss their connection to conformal mappings from the mathematical viewpoint, and then proceed with algorithms and applications.

Circle Packings and discrete conformal mappings. The idea of *circle packing* is the following: Consider non-overlapping circles in a two-dimensional domain, take centers as vertices and put edges whenever two circles touch. If this graph is a triangulation, we have a circle packing. The natural correspondence between two combinatorially equivalent packings is a *discrete conformal mapping.* A Riemann mapping-type theorem states that in the simply connected case all packings can

Figure 16. *Regular Patterns. Left:* The facade of Selfridges, Birmingham (UK) is decorated with a circle pattern. *Right:* The “Kreod” pavilions, London, are derived from the support structure defined by a triangle mesh with incircle packing property.

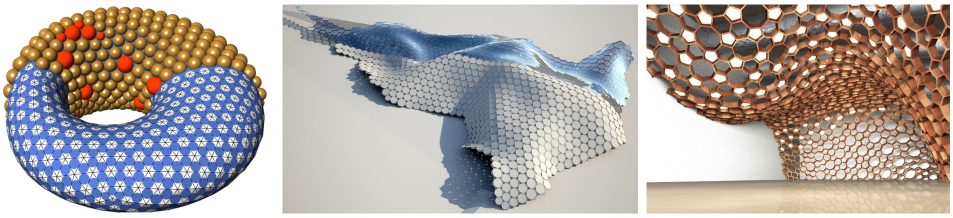


Figure 17. *Left:* An incircle-packing (CP) mesh and its dual ball packing. *Center:* An existing freeform skin (Fiera di Milano) covered by a packing of circles dual to the incircle packing of Lemma 4.2, with circles having radius $\approx r_i$ [35]. *Right:* Torsion-free support structure made of the contact planes in the ball packing.

be conformally mapped to packings which fill either the disk or the unit sphere [38]. The proper statement is the following:

Theorem 4.1 (Koebe-Andreev-Thurston). *Let K be a complex triangulating a compact surface S (possibly with boundary). Then there exists a unique Riemann surface S_K homeomorphic to S and a circle packing with contact graph K which is unique up to conformal automorphisms of S_K and which univalently fills S_K.*

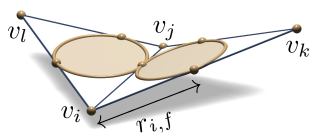
This result (due to Koebe 1936, and more recently to Andreev and Thurston as well as Beardon and Stephenson [3]) justifies the definition of discrete conformal mappings together with the fact that they converge to smooth conformal mappings. We extend the concept of circle packing, starting with an elementary lemma [35]:

Lemma 4.2. *For a triangle mesh (V, E, F) in $\mathbb{R}^n$, the following are equivalent:*

(1) *For faces $f = ijk$, $f' = ijl$, the incircles of triangles $v_i v_j v_k$, $v_i v_j v_l$ have the same contact point with $v_i v_j$ ("incircle packing property").*

(2) *Under the same assumptions, edgelengths obey $|v_i - v_k| + |v_j - v_l| = |v_k - v_j| + |v_i - v_l|$.*

(3) *There are radii $r : V \to [0, \infty)$ with $r_i + r_j = |v_i - v_j|$ for all $ij \in E$.*



Proof. Using the notation of (1), define $r_{i,f}$ as the distance from v_i to the points where the incircle of f touches segments $v_i v_j$ and $v_i v_k$. Obviously, $|v_i - v_j| = r_{i,f} + r_{j,f}$. From $|v_i - v_j| = r_{i,f'} + r_{j,f'}$ we get $r_{i,f} - r_{i,f'} = r_{j,f'} - r_{j,f}$.

Since (1) is expressed as $r_{i,f} = r_{i,f'}$, and (2) is expressed as $r_{i,f} - r_{i,f'} = r_{j,f} - r_{j,f'}$, the equivalence (1) $\Longleftrightarrow$ (2) follows. Further, (1) $\Longrightarrow$ (3) with $r_i = r_{i,f}$, for any face f containing the vertex i. The implication (3) $\Longrightarrow$ (2) is obvious. $\square$

Circle patterns and discrete conformal mappings. The previous result implies that for a planar triangulation (V, E, F) with the incircle packing (CP) property, circles centered in vertices v_i, having radius r_i yield a packing whose contact graph is (V, E, F) (also the converse is true). These circles together with the original incircles form an *orthogonal circle pattern.* Such patterns (without the requirement of

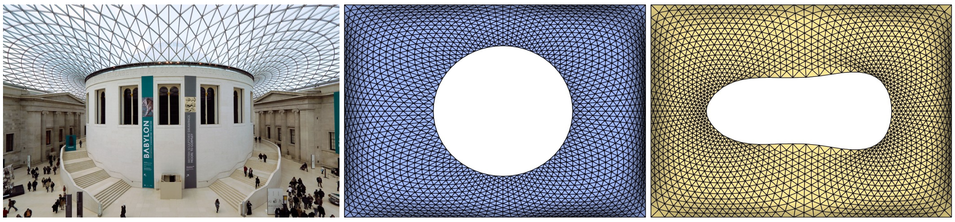


Figure 18. Optimization for the incircle-packing property of meshes which are not simply connected. The triangle mesh of the Great Court of the British Museum is only optimizable if the inner boundary is allowed to move. If one accepts Conjecture 4.3, this is caused by the geometric conformal class of the reference surface not coinciding with the combinatorial conformal class of the mesh.

coming from a triangulation) define discrete conformal mappings in the same way packings do. This discrete approach to conformal mapping has been very fruitful: It has been used by He and Schramm for proving an extended Koebe's theorem [18], and by Bobenko et al. for determining the shape of a minimal surface from the combinatorics of its network of principal curves [5], based on a convergence result by O. Schramm [37].

Circle-Packing Meshes in freeform architecture. Meshes with the CP property of Lemma 4.2 are relevant to architecture in various ways. One reason is aesthetics, since such meshes tend to be regular in the sense of even distribution of angles and edgelengths [35].

Another application is geometric. By Lemma 4.2, the balls of radius r_i, centered in vertices v_i, intersect the incircles orthogonally, and the contact planes of balls constitute a *torsion-free support structure* associated with a mesh (V^*, E^*, F^*) whose vertices are incenters (Figure 17).

In [35], we optimized a triangle mesh for the CP property by moving its vertices tangential to it. This succeeds for meshes of disk and sphere topology, but not for others, see Figure 18. This behaviour can be explained by the following

Conjecture 4.3. *The canonical correspondence between combinatorially equivalent "incircle packing" meshes is a discrete version of conformal equivalence of general surfaces — in much the same way the correspondence between orthogonal circle patterns is a discrete version of conformal equivalence of Riemann surfaces.*

We do not attempt to give a precise statement here. If the conjecture is true, then the combinatorics of a triangle mesh $M = (V, E, F)$ define a "combinatorial" conformal class $[M]$, containing all CP meshes combinatorially equivalent to M. The shapes of those meshes are approximately related by continuous conformal equivalence. If the shape of M is not in $[M]$, there is no CP mesh combinatorially equivalent to M. Since for topological spheres and topological disks there is only one conformal class, optimization for the CP property works in these cases. So far, numerical evidence supports the conjecture, see also Figure 18.

5. Geometry and Statics

Statics is a very extensive and important field in building construction, and it is not surprising that it has connections to geometry. We discuss two specific topics here: One is the effort to include rudimentary statics in computational design (see §6). The other one is the geometry of freeform masonry, see Figure 19.

Self-supporting surfaces. We employ a simplified material model justified by the fact that failure of such structures is via geometry catastrophe, not by material failure. It is called Heyman's safe theorem [20] and postulates stability of a masonry shell S if there is a mesh whose vertices and edges are contained in S and whose edges carry compressive forces which are in equilibrium with the deadload. The geometry of such force systems has been investigated since J. C. Maxwell's time. P. Ash et al. in [1] summarize their geometry and connection to polyhedral bowls (see Figure 19). The use of such force systems ("thrust networks") in architectural design has been pioneered by P. Block, see e.g. [4]. The method is further justified by the interpretation of force networks and polyhedral bowls as finite element discretizations of the shell equations and associated Airy potentials [16].

Discrete differential geometry of masonry shells. In [41] we established a differential-geometric context of freeform masonry together with algorithms for finding the nearest self-supporting shape to a given shape. The discrete shell equations have an interpretation in the framework of curvatures of polyhedral surfaces (§2.4) after a certain duality is applied, and there is also an interpretation in terms of geometric graph Laplacians. One result is the following: If a planar quad mesh is to have equilibrium forces in its edges, it must be aligned with the principal curvature lines in the sense of "relative" differential geometry, with the Airy potential as unit sphere. This is another instance of a problem which has a unique solution, effectively hindering the design process. The design and assembly of self-supporting masonry surfaces is still an active topic of research from both the theory and practice side. We exemplarily point to [17] and [13].

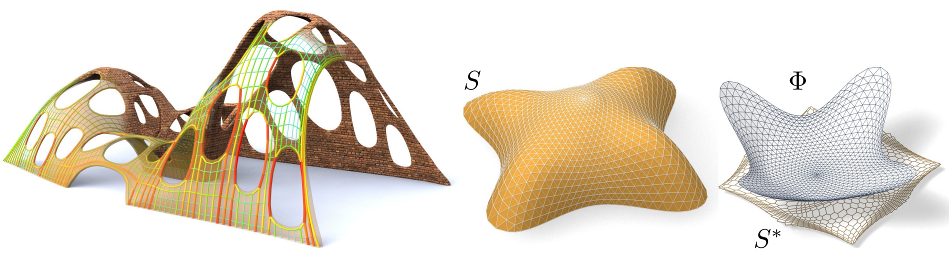


Figure 19. *Self-supporting surfaces.* A shell built from bricks without mortar in theory is stable, if it contains a fictitious network of compressive forces which is in equilibrium with the deadload. *Right:* Such a force system S has an associated polyhedral "Airy potential" bowl Φ (at least locally) whose face gradients are exactly Maxwell's force diagram S^* which is an orthogonal dual to the common projection of both S and Φ onto the horizontal plane.

6. Computational Design

Interactive design often requires solving a large number of nonlinear constraints quickly. An example of that is design of meshes with planar faces. Another one is the inclusion of force systems during design, which can give a user important feedback (without replacing a full-blown statics analysis). A more complex example, not even discrete, is design of *developable* surfaces and curved-folding objects (Figure 20), where constraint equations are found by using spline surfaces and expressing geometric side conditions in terms of the spline control elements [39].

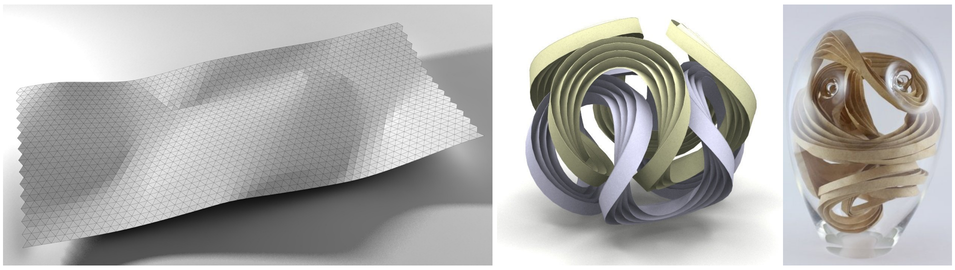


Figure 20. *Design with nonlinear constraints. Left:* A triangle mesh with equilateral faces approximates the "flying carpet" surface of Figure 7; it is a discrete model of crumpled paper [23]. *Center:* A union of developable strips whose developments fit together models a curved-crease sculpture capable of being folded from an annulus [39]. This model is inspired by sculptures by E. and M. Demaine (right). In both situations efficient design uses appropriately regularized Newton methods applied to quadratic equations.

Setting up and solving constraints. A system of n constraints on m variables can be written as $f(x) = 0$, with $x \in \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^n$. An iterative Newton method would have to solve $x_{j-1} + df_{x_{j-1}}(x_j - x_{j-1}) = 0$. Typically this linear system is under-determined (otherwise there would be no design freedom) and also contains redundant equations. These numerical difficulties can be overcome by appropriate regularization which geometrically amounts to "guided" projection onto the constraint manifold in $\mathbb{R}^m$. It turns out that *quadratic* equations give much better convergence than higher order ones, which is in part easy to achieve by introducing more variables. For some constraints like approximation of a reference surface there are special geometric ways to replace them with quadratic terms. For more details we refer to [40].

We see this line of research as a contribution to the longtime goal of interactive design tools capable of handling geometric constraints combined with statics.

Conclusion. We have reported on research in mathematics (discrete differential geometry) and computer science (geometry processing) which is either relevant to free forms in architecture, or is inspired by the geometric problems which occur there. This new area of applications is not exhausted yet; it continues to yield interesting problems and solutions. We emphasize that this field of applications has certain strange features: for artistic reasons it may happen that unique solutions to mathematical problems are are not acceptable in practice. Of course, this "design dilemma" is not restricted to architecture.

The research we reported on in this paper is focussed on architecture. Many ascpects of it are very relevant to other fields of application. This in particular applies to computational design, which lies at the *interface between technology and art*. The newly established doctoral college for computational design at Vienna University of Technology aims at addressing future challenges in this broad field.

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