SPHERICITY AND GEOMETRIC PROPERTIES OF RATIOS IN MÖBIUS AND LAGUERRE GEOMETRY

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Abstract. Motivated by the recent need to work with spherical vertex stars in applications and theory, we contribute to the algebraic description of the sphericity of points and planes. Driven by the well-known characterization of four concyclic points through cross-ratios we extend this notion to sphericity of five points in 3-space by making use of quaternionic ratios which we call diagonal-ratios. We investigate the dual setting and obtain properties of a corresponding diagonal-ratio in terms of dual quaternions for five planes in tangential contact with a sphere.

Keywords: Sphere geometries, cross-ratio, sphericity, dual quaternions.

1. Introduction and Preliminaries

1.1. **Introduction.** In recent years several applications [7, 6, 4] have brought up the necessity to work with spherical vertex stars. A *vertex star* is a vertex (the central vertex) of a mesh together with its edge neighbors. It is called *spherical* if these vertices lie on a common sphere. In a mesh with $\mathbb{Z}^2$ combinatorics a vertex star consists of a central vertex and four neighboring vertices. These five points generically do not lie on a common sphere as a sphere in $\mathbb{R}^3$ is determined by four points in general position.

There are certainly many ways to determine whether five points lie on a common sphere or not. In our paper we generalize the method known for circles. For that we would like to recall the very elegant characterization of four points in a plane (or in space) lying on a common circle. Four points lie on a common circle if and only if their complex or quaternionic cross-ratio is real (cf. [1, 3, 10] and Section 1.2).

In the same way as complex numbers are often used in plane geometry, the skew field of quaternions is brought up to work in three-dimensional space. For that, the three-dimensional imaginary part of the quaternions can be identified with $\mathbb{R}^3$. We use this representation of points and vectors in $\mathbb{R}^3$ to introduce a novel ratio, analogous to the cross-ratio, which involves the diagonals of a pentagon (hence the name *diagonal-ratio* in Definition 6). It turns out that this ratio is independent of reversing their order if and only if the five points lie on a common sphere (Theorem 8).

Points lying on a sphere is invariant under Möbius transformations which map spheres and planes to spheres and planes (potentially swapping some of them). However, neither the cross-ratio nor the diagonal-ratio per se are Möbius invariant. Still some geometric information encoded in these ratios is invariant under those transformations (Lemma 5, Theorem 8).

We are also interested in some kind of a dual point of view. Instead of points on spheres we study planes in tangential contact with spheres. For that we take advantage of descriptions that have been used to work with the special orthogonal group $SO(3)$. The action of its elements is easily representable by quaternion multiplication. This method led to the elegant description of

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all Euclidean motions with the use of dual quaternions, particularly in kinematic geometry [8, 2]. After identifying the set of oriented planes in $\mathbb{R}^3$ with a subset of the dual quaternions (the special unit dual quaternions) we interpret the shape of the cross-ratio of four planes in geometric terms (Theorem 15). If five planes are in tangential contact with a sphere then the diagonal-ratio is independent of reversing the order of planes (Theorem 21).

Planes being in oriented tangential contact with oriented spheres is invariant under so called Laguerre transformations which map spheres and points to spheres and points (potentially swapping some of them). The cross-ratio of four planes in terms of dual quaternions is not Laguerre invariant, however, the geometric meaning of the cross-ratio is Laguerre invariant (Theorem 15) and analogously for the diagonal-ratio.

1.2. **Preliminaries.** We start by setting the notation, then quickly recall quaternions, basic ideas of Möbius and Laguerre geometry and define the notion of the cross-ratio.

1.2.1. *Complex numbers and cross-ratio.* It is common in two-dimensional Möbius geometry to identify the plane with the complex numbers. Their real and imaginary parts are the x- and y-coordinates of the corresponding points

$$(x, y) \in \mathbb{R}^2 \longleftrightarrow x + iy \in \mathbb{C}.$$

We will therefore abuse notation and write for a point p dependent on the setting either $p = (x, y)$ or $p = x+iy$. We further follow the common strategy to extend the complex plane $\mathbb{C}$ conformally by adding ∞ and denote $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$. The Möbius and anti-Möbius transformations

$$z \mapsto \frac{az + b}{cz + d}, \quad \text{and} \quad z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d}, \quad \text{with} \quad a, b, c, d \in \mathbb{C} \quad \text{s.t.} \quad ad - bc \neq 0,$$

act on $\hat{\mathbb{C}}$ sharply 3-transitive. The *cross-ratio* of four pairwise distinct points or complex numbers $a, b, c, d \in \mathbb{C}$ reads

$$\operatorname{cr}(a, b, c, d) = (a - b)(b - c)^{-1}(c - d)(d - a)^{-1}. \tag{1}$$

It is a complex number that is invariant under Möbius transformations but not under anti-Möbius transformations. Denoting $q := \operatorname{cr}(a, b, c, d)$ the set $\{q, \bar{q}\}$ is invariant under both, Möbius and anti-Möbius transformations. In Möbius geometry, circles and straight lines are considered to be the same and can be mapped to each other by Möbius transformations. The lines are those circles which contain ∞. A simple and well-known geometric property of the cross-ratio is the following (see, e.g., [10, p. 32]).

Theorem 1. *Four points a, b, c, d in $\hat{\mathbb{C}}$ lie on a common circle if and only if the cross-ratio satisfies* $\operatorname{cr}(a, b, c, d) \in \mathbb{R}$.

Remark 2. If $q = \operatorname{cr}(a, b, c, d) \notin \mathbb{R}$, the triplets of points a, b, c and a, c, d determine two *different* circumcircles k_1, k_2. In this case the imaginary part of the normalized cross-ratio $q/|q|$ is the sine of the intersection angle between k_1 and k_2 (see, e.g., [10]).

1.2.2. *Quaternions and cross-ratio.* The skew field of *quaternions* $\mathbb{H}$ can be used to express geometry in three-dimensional space and in particular for three-dimensional Möbius geometry. While quaternions are often defined via imaginary units i, j, k in the form $r + ix + jy + kz \in \mathbb{H}$, with $r, x, y, z \in \mathbb{R}$, we prefer to identify them with elements of $\mathbb{R} \times \mathbb{R}^3$ in the form

$$\mathbb{H} = \{[r, v] \mid r \in \mathbb{R}, v \in \mathbb{R}^3\},$$

where $[r, v]$ with $v = (v_1, v_2, v_3)$ corresponds to $r+iv_1+jv_2+kv_3$. The first component $r = \operatorname{Re} q$ of a quaternion $q = [r, v]$ is called *real part* and the second component $v = \operatorname{Im} q$ is called *imaginary part.* The addition in this notation of $\mathbb{H}$ reads $[r, v]+[s, w] = [r+s, v+w]$, and the multiplication reads $[r, v] \cdot [s, w] = [rs - \langle v, w\rangle, rw + sv + v \times w]$, where $\langle\cdot, \cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^3$ and where $\times$ is the cross product. The *conjugation* of $q = [r, v]$ is defined by $\overline{q} = [r, -v]$ and the square root of the real number $q\bar{q}$ is called *norm* of q and is denoted by $|q| = \sqrt{q\bar{q}}$. For every $q \in \mathbb{H} \setminus \{0\}$ its *inverse* is given by $q^{-1} = \overline{q}/|q|^2$. We denote the purely imaginary quaternions by $\operatorname{Im}\mathbb{H} = \{q \in \mathbb{H} \mid q = [0, v], \text{ with } v \in \mathbb{R}^3\}$. To express points and vectors of $\mathbb{R}^3$ with quaternions we identify $\mathbb{R}^3$ with $\operatorname{Im}\mathbb{H}$ via $p \leftrightarrow [0, p]$. Note that the inverse of an imaginary quaternion q is $q^{-1} = -q/|q|^2$ which will appear frequently in our paper. Furthermore, note that two quaternions commute if and only if their imaginary parts are linearly dependent. The *cross-ratio* of four quaternions $a, b, c, d \in \mathbb{H}$ is defined as

$$\operatorname{cr}(a, b, c, d) = (a - b)(b - c)^{-1}(c - d)(d - a)^{-1},$$

where now the order of the factors is crucial as the quaternions are not commutative. In analogy to Theorem 1, there is the following theorem (see, e.g., [1, 3]).

Theorem 3. *Four points a, b, c, d in $\mathbb{R}^3$ (or in $\operatorname{Im}\mathbb{H}$, resp.) lie on a common circle if and only if the cross-ratio satisfies* $\operatorname{cr}(a, b, c, d) \in \mathbb{R}$.

1.2.3. *Laguerre geometry.* Laguerre geometry is the geometry of oriented (hyper)planes and oriented (hyper)spheres and whose transformations preserve the oriented contact between them. The *orientation of a plane* τ with equation $\langle n, x\rangle + d = 0$ is given by a specified direction of a unit normal vector n. The *oriented distance* of a point x to τ is given by $\operatorname{dist}(x, \tau) = \langle n, x\rangle + d$. The *positive [negative] side* of τ is where the oriented distance is positive [negative]. The *orientation of a sphere* is given by the signed radius r. An oriented sphere is in *oriented contact* with an oriented plane if both, the sphere and the plane touch each other in the Euclidean sense, and if the negatively oriented sphere lies on the positive side of the plane or vice versa (see Figure 1). Two oriented spheres are in *oriented contact* if they are in oriented contact with the same oriented plane in the same point. Two parallel planes are *similarly oriented parallel* if their unit normal vectors are the same.

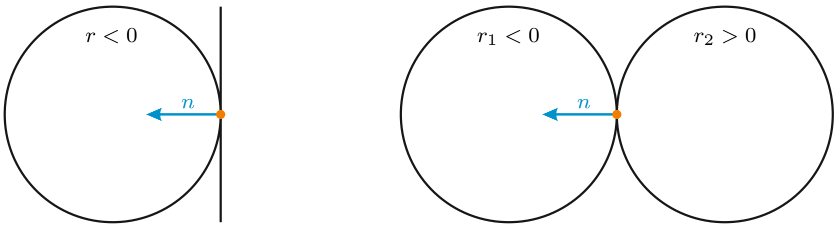

FIGURE 1. Oriented contact. The orientation of circles or spheres is given by the sign of their radius r indicated by clockwise or counterclockwise oriented arrows. An oriented sphere is in *oriented contact* with an oriented plane if they touch and if the negatively oriented sphere lies on the positive side of the plane or vice versa. Two oriented spheres are in *oriented contact* if they are in oriented contact with the same oriented plane in the same point.

2. Sphericity in Möbius Geometry

Three distinct points in a plane always lie on a unique circle (straight lines are also considered as circles). The property that four points lie on a circle is characterized by the property that their cross-ratio is real. Four distinct points in space always lie on a sphere. This *circumsphere* is unique if the four points do not lie on a common circle in which case there are infinitely many spheres containing that circle and therefore the four points.

In Theorem 8 we will characterize when five points in space lie on a common sphere. Our characterization is also in terms of certain ratios depending on imaginary quaternions representing the points. To show that criterion we need some preparatory notions and properties. The following two lemmas can be found in [5, 9], but we include a proof here since it is key for what follows.

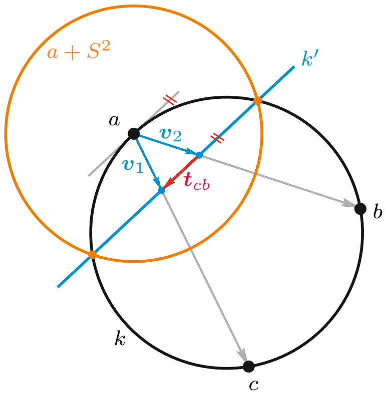

FIGURE 2. Illustration for the proof of Lemma 4: We reflect two points b, c in a circle around a. The connecting line of the reflected points is parallel to the tangent of the circumcircle of a, b, c at point a.

Lemma 4. *Let $a, b, c \in \operatorname{Im}\mathbb{H}$ be pairwise distinct. The imaginary quaternion*

$$t_{cb} := (a-b)^{-1}(b-c)(c-a)^{-1},$$

interpreted as a vector in $\mathbb{R}^3$, is the tangent vector to the circumcircle through a, b, c at point a (cf. Figure 2).

Proof. We compute

$$\begin{aligned} t_{cb} &= (a-b)^{-1}(b-c)(c-a)^{-1} = (b-a)^{-1}(b-a+a-c)(a-c)^{-1} \\ &= \big((b-a)^{-1}(b-a) + (b-a)^{-1}(a-c)\big)(a-c)^{-1} = (a-c)^{-1} + (b-a)^{-1} = v_1 - v_2, \end{aligned}$$

where we set $v_1 := (a-c)^{-1}$ and $v_2 := (a-b)^{-1}$. We therefore obtain $t_{cb} = v_1 - v_2$. Since $v_1, v_2 \in \operatorname{Im}\mathbb{H}$ we have $v_1 = \frac{c-a}{|c-a|^2}$ and $v_2 = \frac{b-a}{|b-a|^2}$ which are reflections of $c-a$ and $b-a$ in the unit sphere S^2 (cf. Figure 2). Consequently, the reflection in the unit sphere with center a maps b to $a + v_2$ and c to $a + v_1$.

The reflection in the sphere $a + S^2$ maps the circumcircle k through a, b, c to a line k' since a is mapped to ∞. By symmetry reasons k' is parallel to the tangent of k in a. Furthermore, this line k' is parallel to t_{cb} which proves the lemma. □

Note that the proof above implies that t_{cb} can be written in the form

$$t_{cb} = (a-c)^{-1} + (b-a)^{-1},$$

which immediately implies

$$t_{cb} = -t_{bc}. \tag{2}$$

Furthermore, note that the expression of t_{cb} with reversed inversions in the product, i.e.,

$$\tilde{t}_{cb} := (a-b)(b-c)^{-1}(c-a)$$

is parallel to t_{cb} since $\tilde{t}_{cb}^{-1} = -(a-c)^{-1}(c-b)(b-a)^{-1} = -(a-b)^{-1} - (c-a)^{-1} = t_{cb}$. Consequently,

$$\tilde{t}_{cb} = \frac{-t_{cb}}{|t_{cb}|^2}. \tag{3}$$

Throughout the paper, we will heavily use this notation also in connection with other points but always rooted at a, for example,

$$t_{dc} = (a-c)^{-1}(c-d)(d-a)^{-1}.$$

Lemma 5. *For $a, b, c, d \in \operatorname{Im}\mathbb{H}$ the vector $\operatorname{Im}\operatorname{cr}(a, b, c, d)$ represents the normal vector to the circumsphere through a, b, c, d at point a.*

Proof. Using Lemma 4 and Equation (3) we get

$$\begin{aligned} \operatorname{cr}(a,b,c,d) &= (a-b)(b-c)^{-1} \cdot 1 \cdot (c-d)(d-a)^{-1} = \\ &= -(a-b)(b-c)^{-1} \cdot (c-a)(a-c)^{-1} \cdot (c-d)(d-a)^{-1} = \\ &= -\tilde{t}_{cb} t_{dc} = \frac{1}{|t_{cb}|^2} t_{cb} t_{dc} = \frac{1}{|t_{cb}|^2}[-\langle t_{cb}, t_{dc}\rangle, t_{cb} \times t_{dc}]. \end{aligned} \tag{4}$$

Therefore, $\operatorname{Im}\operatorname{cr}(a,b,c,d) = \frac{1}{|t_{cb}|^2}(t_{cb} \times t_{dc})$. Lemma 4 implies that t_{cb} is a tangent vector to the circumcircle of a, b, c at point a, and analogously, t_{dc} is a tangent vector to the circumcircle of a, c, d at point a, they both lie in the tangent plane to the circumsphere of a, b, c, d in a. Hence, their cross product is parallel to the normal vector at point a. □

With Lemma 5 we obtain an alternative proof of Theorem 3: Let k_1, k_2 be the circumcircles of a, b, c and a, c, d, respectively. Both lie on the circumsphere of a, b, c, d. Equation (4) implies that the cross-ratio $\operatorname{cr}(a, b, c, d)$ is real if and only if the cross product $t_{cb} \times t_{dc}$ of the tangent vectors in a to k_1 and k_2 vanishes. Consequently, the cross-ratio is real if and only if t_{cb} and t_{dc} are parallel which means that k_1 and k_2 coincide.

With the following definition we "generalize" the cross-ratio of four points to five points by including the diagonals of the pentagon.

Definition 6. *Let $a, b, c, d, e \in \operatorname{Im}\mathbb{H}$ be five distinct points. We call the following ratio of edges and diagonals*

$$\operatorname{dr}(a,b,c,d,e) := (a-b)(b-e)^{-1}(e-a)(a-d)^{-1}(d-e)(e-c)^{-1}(c-d)(d-b)^{-1}(b-c)(c-a)^{-1}$$

the diagonal-ratio *of a, b, c, d, e (cf. Figure 3).*

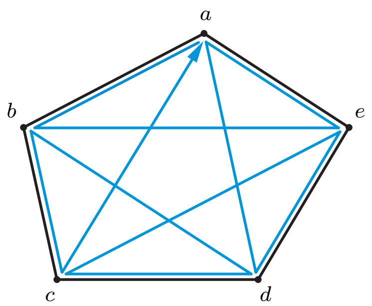

FIGURE 3. Illustration of the vectors involved in the diagonal-ratio.

Lemma 7. *Let $a, b, c, d, e \in \operatorname{Im}\mathbb{H}$ be five distinct points. Then the diagonal-ratio factorizes into cross-ratios*

$$\operatorname{dr}(a,b,c,d,e) = \operatorname{cr}(a,b,e,d) \cdot \operatorname{cr}(a,e,c,d) \cdot \operatorname{cr}(a,d,b,c). \tag{5}$$

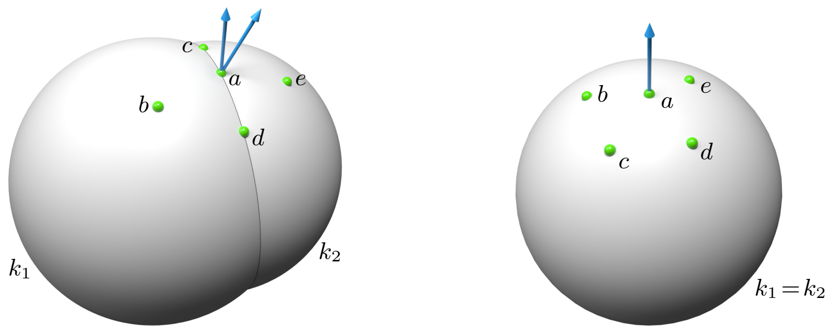

FIGURE 4. *Left:* The five points a, b, c, d, e do not lie on a common sphere. The quadruplet a, b, c, d defines sphere k_1 and the quadruplet a, c, d, e defines sphere k_2. The two spheres intersect along the circle through a, c, d and their normal vectors are linearly independent. *Right:* All five points are co-spherical.

Proof. In analogy to Lemma 5, we express the diagonal-ratio in terms of tangent vectors at a

$$\begin{aligned}
\operatorname{dr}(a,b,c,d,e) &= (a-b)(b-e)^{-1}(e-a)(a-d)^{-1}(d-e) \\
&\quad (e-c)^{-1}(c-d)(d-b)^{-1}(b-c)(c-a)^{-1} \\
&= \big((a-b)(b-e)^{-1}(e-a)\big)\big((a-d)^{-1}(d-e)(e-a)^{-1}\big) \\
&\quad \big((a-e)(e-c)^{-1}(c-a)\big)\big((a-c)^{-1}(c-d)(d-a)^{-1}\big) \\
&\quad \big((a-d)(d-b)^{-1}(b-a)\big)\big((a-b)^{-1}(b-c)(c-a)^{-1}\big) \\
&= \tilde{t}_{eb}t_{ed}\tilde{t}_{ce}t_{dc}\tilde{t}_{bd}t_{cb} = -\frac{1}{|t_{eb}|^2}t_{eb}t_{ed}\frac{1}{|t_{ce}|^2}t_{ce}t_{dc}\frac{1}{|t_{bd}|^2}t_{bd}t_{cb}.
\end{aligned} \tag{6}$$

Using Equations (4) and (2) concludes the proof. □

The following theorem characterizes five points lying on a sphere in terms of this diagonal-ratio. We assume the elementary geometric fact that two spheres passing through the same circle are identical if and only if the two normal vectors of the spheres in a common point of the circle are parallel.

Theorem 8. *Five points $a, b, c, d, e \in \operatorname{Im}\mathbb{H}$ lie on a common sphere if and only if*

$$\operatorname{dr}(a,b,c,d,e) = \operatorname{dr}(a,e,d,c,b).$$

Then both $\operatorname{Im}\operatorname{dr}(a,b,c,d,e)$ and $\operatorname{Im}\operatorname{dr}(a,e,d,c,b)$, respectively, represent a normal vector of the circumsphere at point a.

Proof. In general the two quadruples of points a, b, c, d and a, c, d, e lie on two different spheres k_1 and k_2, both containing a, c, and d, and therefore their circumcircle (see Figure 4).

We have to show that k_1 and k_2 coincide if and only if $\operatorname{dr}(a,b,c,d,e) = \operatorname{dr}(a,e,d,c,b)$. We first express both diagonal-ratios in terms of tangent vectors at point a via (6)

$$\begin{aligned}
\operatorname{dr}(a,b,c,d,e) &= \frac{1}{|t_{be}|^2}t_{be}t_{ed}\frac{1}{|t_{ec}|^2}t_{ec}t_{dc}\frac{1}{|t_{db}|^2}t_{db}t_{cb}, \\
\operatorname{dr}(a,e,d,c,b) &= \frac{1}{|t_{eb}|^2}t_{eb}t_{bc}\frac{1}{|t_{bd}|^2}t_{bd}t_{cd}\frac{1}{|t_{ce}|^2}t_{ce}t_{de}.
\end{aligned}$$

Since $t_{be} = -t_{eb}$ etc. and by setting $\alpha := 1/|t_{be}t_{ec}t_{db}|^2$ we obtain

$$\begin{aligned}
\operatorname{dr}(a,b,c,d,e) - \operatorname{dr}(a,e,d,c,b) &= \alpha t_{be}t_{ed}t_{ec}t_{dc}t_{db}t_{cb} - \alpha t_{eb}t_{bc}t_{bd}t_{cd}t_{ce}t_{de} \\
&= \alpha t_{be}(t_{ed}t_{ec}t_{dc}t_{db}t_{cb} - t_{cb}t_{db}t_{dc}t_{ec}t_{ed}).
\end{aligned}$$

Expanding yields

$$\alpha t_{be}\big([-\langle t_{ed},t_{ec}\rangle, t_{ed}\times t_{ec}]\cdot t_{dc}\cdot[-\langle t_{db},t_{cb}\rangle, t_{db}\times t_{cb}]$$
$$-[-\langle t_{cb},t_{db}\rangle, t_{cb}\times t_{db}]\cdot t_{dc}\cdot[-\langle t_{ec},t_{ed}\rangle, t_{ec}\times t_{ed}]\big).$$

After setting $r_1 := -\langle t_{db},t_{cb}\rangle$, $r_2 := -\langle t_{ed},t_{ec}\rangle$, $n_1 := t_{cb}\times t_{db}$, $n_2 := t_{ed}\times t_{ec}$, and $t := t_{dc}$, the term simplifies to

$$\alpha t_{be}\big([r_2,n_2]\cdot t\cdot[r_1,-n_1]-[r_1,n_1]\cdot t\cdot[r_2,-n_2]\big).$$

Inserting the identity

$$t\cdot[r_l,-n_l]\cdot t=[\langle t,n_l\rangle, r_l t-(t\times n_l)]\cdot t=[-r_l\langle t,t\rangle,\langle t,n_l\rangle t-(t\times n_l)\times t]=-|t|^2[r_l,n_l] \tag{7}$$

in the above we get

$$\mathrm{dr}(a,b,c,d,e)-\mathrm{dr}(a,e,d,c,b)=-\alpha t_{be}|t|^2\big([r_2,n_2][r_1,n_1]-[r_1,n_1][r_2,n_2]\big)\cdot t^{-1}.$$

We see that $\mathrm{dr}(a,b,c,d,e)-\mathrm{dr}(a,e,d,c,b)=0$ if and only if the term in the bracket is zero. This is the case if and only if the product of the quaternions $[r_1,n_1]$ and $[r_2,n_2]$ commutes, i.e., if and only if their imaginary parts n_1 and n_2 are linearly dependent. Since n_1 and n_2 are normal vectors at point a to the spheres k_1 and k_2, respectively, it means that k_1 and k_2 coincide if and only if n_1 and n_2 are linearly dependent which proves the first part of the theorem.

To prove the second part we recall Equation (6) and expand it

$$\begin{aligned}\mathrm{dr}(a,b,c,d,e)&=\alpha t_{be}t_{ed}t_{ec}t_{dc}t_{db}t_{cb}\\&=\alpha[-\langle t_{be},t_{ed}\rangle, t_{be}\times t_{ed}]\,[-\langle t_{ec},t_{dc}\rangle, t_{ec}\times t_{dc}]\,[-\langle t_{db},t_{cb}\rangle, t_{db}\times t_{cb}].\end{aligned} \tag{8}$$

The three quaternions in this latter product all have vector-parts parallel to the normal vector of the sphere $k_1=k_2$ at point a. Since the product of quaternions with linearly dependent imaginary parts yields a quaternion with an imaginary part linearely dependent to the previous ones, $\mathrm{Im}\,\mathrm{dr}(a,b,c,d,e)$ must also be parallel to the normal vector of $k_1=k_2$ at point a. □

Corollary 9. *If five points a,b,c,d,e in $\mathbb{R}^3$ lie on a common circle then* $\mathrm{dr}(a,b,c,d,e)\in\mathbb{R}$.

Proof. If the points a,b,c,d,e lie on the same circle then each four of them are also concyclic. Therefore the cross-ratios in Equation (5) are real, hence so is $\mathrm{dr}(a,b,c,d,e)$. □

Remark 10. A real diagonal-ratio however does not imply that the five points lie on a circle as the following example shows. Choosing

$$\begin{aligned}&a=(1,0,0),\quad b=(0,1,0),\quad c=(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2},0),\quad d=(-1,0,0),\\&e=\big(\tfrac{5}{8}(\sqrt{3}-2),-\tfrac{1}{8}\sqrt{100\sqrt{3}-159},0\big),\end{aligned}$$

yields

$$\mathrm{dr}(a,b,c,d,e)=[\tfrac{5}{2892}(60+85\sqrt{3}-\sqrt{36228\sqrt{3}-38349}),0]\in\mathbb{R}$$

however

$$\mathrm{cr}(a,b,e,d)=[\tfrac{1}{1928}(539-100\sqrt{3}+5\sqrt{12076\sqrt{3}-12783}),\big(0,0,\frac{3}{2(5+\sqrt{100\sqrt{3}-159})}\big)]\notin\mathbb{R}.$$

Consequently, a,b,d,e are not concyclic.

3. Sphericity in Laguerre Geometry

The algebraic structure often used to describe elements and transformations in (planar) Laguerre geometry is the ring of *dual numbers* $\mathbb{D}$ defined as (see, e.g., [10])

$$\mathbb{D} = \{x + \epsilon y \mid x, y \in \mathbb{R}\},$$

where $\epsilon \neq 0$ is a symbol satisfying $\epsilon^2 = 0$ and where x and y are called *real* and *dual part*, respectively. Adding dual numbers works componentwise $(a + \epsilon b) + (c + \epsilon d) = (a + c) + \epsilon(b + d)$ and the product of dual numbers reads $(a + \epsilon b) \cdot (c + \epsilon d) = ac + \epsilon(ad + bc)$.

The *conjugate* of $z = a + \epsilon b$ is $\bar{z} = a - \epsilon b$ and its *modulus* is defined as the square root of $z\bar{z} = a^2$ and therefore equals $|a|$. Consequently, a dual number has an *inverse* if and only if its modulus does not vanish in which case it is $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a-\epsilon b}{a^2} = \frac{1}{a} - \epsilon \frac{b}{a^2}$.

An oriented straight lines in the plane which is not parallel to the x-axis can be identified by its *polar coordinates* (ϕ, s), where ϕ denotes the oriented angle with the x-axis and where s denotes the oriented distance of its intersection point with the x-axis to the origin. Consequently, oriented straight lines can be identified with dual numbers in the following way

$$(\phi, s) \in ((0, 2\pi) \setminus \pi) \times \mathbb{R} \longleftrightarrow \tan\big(\frac{\phi}{2}\big)(1 + \epsilon s) \in \mathbb{D}.$$

For lines parallel to the x-axis this identification does not work and it is necessary to assign adapted dual numbers to them (for details, see [10, p. 81-83]).

Theorem 11 ([10, p. 93-94])**.** *Four oriented lines in $\mathbb{R}^2$ touch a common oriented circle if and only if the corresponding dual numbers $a, b, c, d \in \mathbb{D}$ satisfy* $\mathrm{cr}(a, b, c, d) \in \mathbb{R}$ *where the cross-ratio definition has the same algebraic composition as for complex numbers in Equation* (1)*.*

Remark 12. For the generic case, $q = \mathrm{cr}(a, b, c, d) \notin \mathbb{R}$ the triplets of oriented lines a, b, c and a, c, d determine two *different* circles k_1, k_2. In this case the imaginary part of the normalized cross-ratio $q/|q|$ is the tangent distance between k_1 and k_2 [10, p. 162].

To describe oriented planes in space, we choose a subset of the *dual quaternions*

$$\mathbb{DH} = \{aq \mid a \in \mathbb{D}, q \in \mathbb{H}\} = \{q + \epsilon p \mid p, q \in \mathbb{H}\},$$

which form a skew ring. Dual quaternions are often used to describe motions in kinematic geometry. For a dual quaternion $d = q + \epsilon p \in \mathbb{DH}$, we call q its *quaternion-part* and p its *dual part*. We will also sometimes refer to $\operatorname{Im} q$ as *vector-part*.

In analogy to dual numbers, a dual quaternion $d = q + \epsilon p$ with $q \neq 0$ has an inverse

$$d^{-1} = q^{-1}(1 - \epsilon p q^{-1})$$

since $dd^{-1} = (q + \epsilon p)(q^{-1}(1 - \epsilon p q^{-1})) = qq^{-1} + \epsilon p q^{-1} - \epsilon q q^{-1} p q^{-1} - \epsilon^2 p q^{-1} p q^{-1} = 1$. The inverse of the product of dual quaternions $d_1, d_2 \in \mathbb{DH}$ is the reversed product of their inverses $(d_1 d_2)^{-1} = d_2^{-1} d_1^{-1}$.

For our purpose, we will need dual quaternions with a purely imaginary unit quaternion-part and real dual part which we denote by

$$\mathcal{S} := \{q + \epsilon r \in \mathbb{DH} \mid q \in \operatorname{Im} \mathbb{H}, |q| = 1 \text{ and } r \in \mathbb{R}\}$$

and which we call *special unit dual quaternions*. Clearly, special unit dual quaternions are invertible.

Each oriented plane in $\mathbb{R}^3$ can be represented by an affine equation $\langle n, x\rangle + d = 0$ with unit normal vector n and oriented distance to the origin d. The two equations

$$\langle n, x\rangle + d = 0 \quad \text{and} \quad \langle -n, x\rangle - d = 0$$

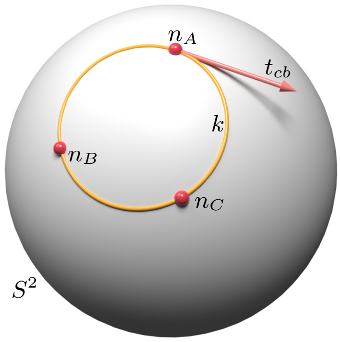

FIGURE 5. The end points of the three vectors n_A, n_B, n_C lie on the unit sphere S^2. The tangent vector t_{cb} of their circumcircle k can be computed via quaternion product (Lemma 13).

represent the same Euclidean plane but they are oriented differently. Two parallel planes are *similarly oriented parallel* if their unit normal vectors are the same. We identify each oriented plane with a special unit dual quaternion by

$$\langle n, x\rangle + d = 0 \longleftrightarrow [0, n] + \epsilon d \in \mathcal{S}.$$

From here on we will frequently omit the word "*oriented*" for oriented spheres, oriented planes or oriented contact, since in Laguerre geometry we always deal with oriented objects and make it a general assumption.

Lemma 13. *Let*

$$a = [0, n_A] + \epsilon d_A, \quad b = [0, n_B] + \epsilon d_B, \quad c = [0, n_C] + \epsilon d_C \in \mathcal{S}$$

represent three planes A, B, C which are pairwise not similarly oriented parallel. Furthermore, let the circle through the three points n_A, n_B, n_C on the unit sphere S^2 be denoted by k. Then the vector-part of the dual quaternion

$$s_{cb} := (a-b)^{-1}(b-c)(c-a)^{-1}$$

represents the tangent vector t_{cb} to the circle k at point n_A (see Figure 5).

Proof. Since A and B are not similarly oriented parallel planes, the dual number $a - b$ has a non-vanishing quaternion-part and is therefore an invertible dual quaternion. The computation starts the same way as in the proof of Lemma 4 (applied at $(*)$) until we insert the corresponding dual quaternions:

$$\begin{aligned}
s_{cb} &= (a-b)^{-1}(b-c)(c-a)^{-1} \overset{(*)}{=} (a-c)^{-1} + (b-a)^{-1} \\
&= ([0, n_A - n_C] + \epsilon(d_A - d_C))^{-1} + ([0, n_B - n_A] + \epsilon(d_B - d_A))^{-1} \\
&= [0, n_A - n_C]^{-1}(1 - \epsilon(d_A - d_C)[0, n_A - n_C]^{-1}) + [0, n_B - n_A]^{-1}(1 - \epsilon(d_B - d_A)[0, n_B - n_A]^{-1}) \\
&= \frac{1}{|n_A - n_C|^2}\big([0, -n_A + n_C] + \epsilon(d_A - d_C)\big) + \frac{1}{|n_B - n_A|^2}\big([0, -n_B + n_A] + \epsilon(d_B - d_A)\big) \\
&= \Big[0, \frac{n_C - n_A}{|n_C - n_A|^2} + \frac{n_A - n_B}{|n_A - n_B|^2}\Big] + \epsilon\Big(\frac{d_A - d_C}{|n_A - n_C|^2} + \frac{d_B - d_A}{|n_B - n_A|^2}\Big).
\end{aligned}$$

Setting $v_1 := \frac{n_C - n_A}{|n_C - n_A|^2}$ and $v_2 := \frac{n_B - n_A}{|n_B - n_A|^2}$ yields $t_{cb} := v_1 - v_2$ as vector-part of s_{cb}. Following the arguments in the proof of Lemma 4 and Figure 2, the vector-part of s_{cb}, which is $t_{cb} = v_1 - v_2$, is parallel to the tangent line of k at point n_A. □

Note that from the proof above we obtain that s_{cb} can be expressed as

$$s_{cb} = [0, t_{cb}] + \epsilon\lambda_{cb},$$

where $\lambda_{cb} := \frac{d_A - d_C}{|n_A - n_C|^2} + \frac{d_B - d_A}{|n_B - n_A|^2} \in \mathbb{R}$. It also immediately implies $s_{cb} = -s_{bc}$. Furthermore, note that the expression of s_{cb} with reversed inversions in the product, i.e.,

$$\tilde{s}_{cb} := (a-b)(b-c)^{-1}(c-a)$$

has a parallel vector-part to the one of s_{cb} since

$$\tilde{s}_{cb}^{-1} = -(a-c)^{-1}(c-b)(b-a)^{-1} = -s_{bc} = [0, -t_{bc}] - \epsilon\lambda_{bc},$$

Consequently,

$$\tilde{s}_{cb} = [0, -t_{bc}]^{-1}(1 + \epsilon\lambda_{bc}[0, -t_{bc}]^{-1}) = \frac{-1}{|t_{cb}|^2}([0, t_{cb}] + \epsilon\lambda_{bc}). \tag{9}$$

The following lemma is the Laguerre geometric analogue to Lemma 5.

Lemma 14. *Let $a, b, c, d \in \mathcal{S}$ represent four planes A, B, C, D which are not similarly oriented parallel. Then the vector-part of their cross-ratio $\operatorname{cr}(a, b, c, d)$ is parallel to the normal vector n_A of plane A.*

Proof. Using Lemma 13 and Equation (9) we get

$$\begin{aligned} \operatorname{cr}(a, b, c, d) &= (a-b)(b-c)^{-1}(c-d)(d-a)^{-1} \\ &= -(a-b)(b-c)^{-1}(c-a)(a-c)^{-1}(c-d)(d-a)^{-1} \\ &= \frac{1}{|t_{cb}|^2}([0, t_{cb}] + \epsilon\lambda_{bc})([0, t_{dc}] + \epsilon\lambda_{dc}) \\ &= \frac{1}{|t_{cb}|^2}[-\langle t_{cb}, t_{dc}\rangle, t_{cb} \times t_{dc}] + \epsilon\frac{1}{|t_{cb}|^2}[0, \lambda_{dc}t_{cb} + \lambda_{bc}t_{dc}]. \end{aligned} \tag{10}$$

Since t_{cb} and t_{dc} are both tangents in n_A to the circles through $n_A, n_B, n_C \in S^2$ and $n_A, n_C, n_D \in S^2$, respectively, their cross product points in the direction of the normal to the unit sphere in n_A which is parallel to the vector n_A itself, i.e., the normal vector of the plane A. $\square$

Unlike in Möbius geometry, where four points determine a unique sphere (unless they lie on a common circle), in Laguerre geometry four planes are not necessarily in oriented contact with a common sphere. For example if two planes are similarly oriented parallel, there is no oriented sphere in oriented contact with both planes. Another example exists of three planes enveloping a rotational cone and a fourth plane being similarly oriented parallel to a tangent plane of that cone. See Figure 6 (**Case (iii)**) for illustrations of that setting.

Three planes A, B, C which are pairwise not similarly oriented parallel always envelope a rotational cone k (or cylinder which is the same in Laguerre geometry). There are three essentially different possible arrangements of an additional fourth distinct plane D in relation to the three given planes, which are illustrated schematically in Figure 6.

Theorem 15. *Let A, B, C, D be four planes which are pairwise not similarly oriented parallel. Their arrangement in space determines the structure of their cross-ratio. For some $r \in \mathbb{R}, n, v \in \operatorname{Im}\mathbb{H}$ we have*

(i) $\operatorname{cr}(a, b, c, d) = r$ $\Leftrightarrow$ *A, B, C, D touch a rotational cone,*
(ii) $\operatorname{cr}(a, b, c, d) = [r, n] + \epsilon \cdot [0, v]$ *with $n \neq 0$* $\Leftrightarrow$ *A, B, C, D touch a unique sphere,*
(iii) $\operatorname{cr}(a, b, c, d) = [r, 0] + \epsilon \cdot [0, v]$ *with $v \neq 0$* $\Leftrightarrow$ *A, B, C, D touch no common sphere.*

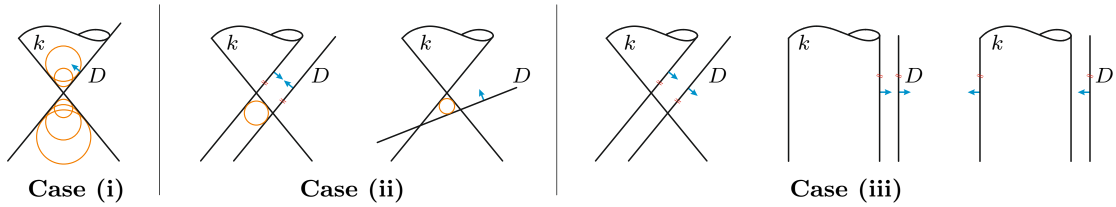

FIGURE 6. Any three planes A, B, C which are pairwise not similarly oriented parallel envelope a rotational cone or cylinder k. There are three essential different positions of a fourth plane D with respect to k. **Case (i):** D is another tangent plane of that cone k in which case there are infinitely many spheres in oriented contact with the four planes. **Case (ii):** D is not similarly parallel to any tangent plane in which case there is exactly one sphere in oriented contact with the four planes. **Case (iii):** D is similarly parallel but distinct to a tangent plane of k in which case there is no sphere in oriented contact with the four planes.

Proof. Before we go through the individual cases we make some general preparations. From Equation (10) we obtain the cross-ratio of four planes in the following form:

$$\operatorname{cr}(a,b,c,d) = \frac{1}{|t_{cb}|^2}[-\langle t_{cb}, t_{dc}\rangle, t_{cb} \times t_{dc}] + \epsilon \frac{1}{|t_{cb}|^2}[0, \lambda_{dc} t_{cb} + \lambda_{bc} t_{dc}].$$

By setting

$$r := \frac{-1}{|t_{cb}|^2}\langle t_{cb}, t_{dc}\rangle, \qquad n := \frac{1}{|t_{cb}|^2} t_{cb} \times t_{dc}, \qquad v := \frac{1}{|t_{cb}|^2}(\lambda_{dc} t_{cb} + \lambda_{bc} t_{dc}),$$

the cross-ratio becomes

$$\operatorname{cr}(a,b,c,d) = [r, n] + \epsilon \cdot [0, v].$$

Since three planes always envelope a rotational cone (or cylinder), we denote the cone enveloped by A, B, C by k_1, and the cone enveloped by A, C, D by k_2. Let s_1 and s_2 be arbitrary spheres which are in oriented contact with k_1 and k_2, respectively. We denote their centers by m_1 and m_2, and their radii by r_1 and r_2. Then the distances of the planes from the origin can be expressed as (see Figure 7):

$$\begin{aligned}
\langle m_1, n_A\rangle + r_1 &= \langle m_2, n_A\rangle + r_2 = -d_A,\\
\langle m_1, n_B\rangle + r_1 & \qquad\qquad\qquad\;\; = -d_B,\\
\langle m_1, n_C\rangle + r_1 &= \langle m_2, n_C\rangle + r_2 = -d_C,\\
& \quad\;\; \langle m_2, n_D\rangle + r_2 = -d_D.
\end{aligned}$$

Consequently,

$$\langle m_1, n_A\rangle - \langle m_2, n_A\rangle = \langle m_1, n_C\rangle - \langle m_2, n_C\rangle = r_2 - r_1 \quad \text{and} \quad \langle m_1 - m_2, n_A - n_C\rangle = 0. \tag{11}$$

For the coefficients $\lambda_{bc}, \lambda_{dc}$ in the dual part of the cross-ratio we obtain

$$\begin{aligned}
\lambda_{bc} &= \frac{d_A - d_B}{|n_A - n_B|^2} + \frac{d_C - d_A}{|n_C - n_A|^2}\\
&= \frac{-\langle m_1, n_A\rangle - r_1 + \langle m_1, n_B\rangle + r_1}{|n_A - n_B|^2} + \frac{-\langle m_1, n_C\rangle - r_1 + \langle m_1, n_A\rangle + r_1}{|n_C - n_A|^2} \qquad (12)\\
&= \frac{\langle m_1, n_B - n_A\rangle}{|n_B - n_A|^2} + \frac{\langle m_1, n_A - n_C\rangle}{|n_A - n_C|^2} = \langle m_1, \frac{n_B - n_A}{|n_B - n_A|^2} + \frac{n_A - n_C}{|n_A - n_C|^2}\rangle = \langle m_1, t_{bc}\rangle,
\end{aligned}$$

and analogously $\lambda_{dc} = \langle m_2, t_{dc}\rangle$. Hence the imaginary part v of the dual part can always be expressed as

$$v = \frac{1}{|t_{cb}|^2}(\langle m_2, t_{dc}\rangle t_{cb} + \langle m_1, t_{bc}\rangle t_{dc}). \tag{13}$$

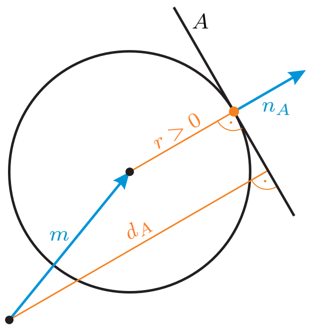

FIGURE 7. Illustration of an oriented sphere with center m and radius r in oriented contact with an oriented plane given by the equation $\langle n_A, x\rangle + d_A = 0$. We can read off the elementary relation $\langle m, n_A\rangle + r = -d_A$.

To prove (i) we first assume that A, B, C, D touch the common rotational cone $k_1 = k_2$. Then the four normal vectors n_A, n_B, n_C, n_D represent four points on a common circle on S^2. Since the two vectors t_{cb} and t_{dc} are tangent vectors to this circle at point n_A (as learned at the end of the proof of Lemma 13), these vectors must be parallel, i.e., $t_{dc} = \mu t_{cb}$ for some $\mu \in \mathbb{R} \setminus 0$. Consequently, we obtain $n = 0$.

Furthermore, since the cones conincide $k_1 = k_2$, we can choose the two spheres s_1, s_2 to be equal $s_1 = s_2$ (every sphere which touches the cone also touches all four planes). Denoting its center by $m := m_1 = m_2$ and using $t_{dc} = \mu t_{cb}$ we obtain for v

$$v = \frac{1}{|t_{cb}|^2}(\langle m_2, t_{dc}\rangle t_{cb} + \langle m_1, t_{bc}\rangle t_{dc}) = \frac{1}{|t_{cb}|^2}(\langle m, \mu t_{cb}\rangle t_{cb} + \langle m, t_{bc}\rangle \mu t_{cb}) = 0,$$

since $t_{bc} = -t_{cb}$. Therefore $\mathrm{cr}(a, b, c, d) = r$.

For the other direction of (i) we assume $\mathrm{cr}(a, b, c, d) = r$. Consequently, $v = 0$ and $n = 0$ which immediately implies $t_{dc} = \mu t_{cb}$ for some $\mu \in \mathbb{R} \setminus 0$ and furthermore

$$0 = v\,|t_{cb}|^2 = \langle m_2, t_{dc}\rangle t_{cb} + \langle m_1, t_{bc}\rangle t_{dc} = \langle m_2, \mu t_{cb}\rangle t_{cb} - \langle m_1, t_{cb}\rangle \mu t_{cb} = \mu\langle m_2 - m_1, t_{cb}\rangle t_{cb},$$

which yields

$$\langle m_2 - m_1, t_{cb}\rangle = \langle m_2 - m_1, t_{dc}\rangle = 0.$$

We therefore have

$$0 = \langle m_2 - m_1, t_{dc}\rangle = \langle m_2 - m_1, \frac{n_A - n_C}{|n_A - n_C|^2} + \frac{n_D - n_A}{|n_D - n_A|^2}\rangle \overset{(11)}{=} \langle m_2 - m_1, \frac{n_D - n_A}{|n_D - n_A|^2}\rangle,$$

which further implies

$$\langle m_2, n_D\rangle - \langle m_2, n_A\rangle = \langle m_1, n_D\rangle - \langle m_1, n_A\rangle.$$

Again using Equation (11) we obtain

$$\langle m_1, n_D\rangle = r_2 - r_1 - r_2 - d_D = -r_1 - d_D,$$

which implies that the sphere s_1 is also in oriented contact with plane D. Analogously, it follows from $0 = \langle m_2 - m_1, t_{cb}\rangle$ that the sphere s_2 is also in oriented contact with plane B. Consequently, all four planes are in oriented contact with two different spheres which implies that they are in tangential contact with a rotational cone.

As for (ii) we first note that $n \neq 0$ is equivalent to t_{cb} and t_{dc} not being parallel. Since t_{cb} is a tangent vector to S^2 touching the circumcircle of n_A, n_B, n_C and t_{dc} is a tangent vector to S^2 touching the circumcircle of n_A, n_C, n_D, both at point n_A, we conclude that this is further equivalent to the four points n_A, n_B, n_C, n_D not lying in a plane. Consequently, any linear system of the form (vectors are notated as columns)

$$\begin{pmatrix} n_A^\top & 1 \\ n_B^\top & 1 \\ n_C^\top & 1 \\ n_D^\top & 1 \end{pmatrix} \begin{pmatrix} m \\ r \end{pmatrix} + \begin{pmatrix} d_A \\ d_B \\ d_C \\ d_D \end{pmatrix} = 0$$

has a unique solution for m and r representing the unique sphere in oriented contact with the four planes given by equations $\langle n_A, x\rangle + d_A = 0$, etc.

Property (iii) follows by eliminating cases (i) and (ii). □

Remark 16. Property (i) of Theorem 15 states that the cross-ratio of four oriented planes is real if and only if they touch a rotational cone (i.e., they touch a one parameter family of spheres). This is the analogue of Theorem 3 of Möbius geometry, stating that the cross-ratio of four points is real if and only if they lie on a common circle, i.e., on a one parameter family of spheres.

The following lemma is the Laguerre analogue to Lemma 7 where the diagonal-ratio definition has the same algebraic composition as for quaternions in Definition 6.

Lemma 17. *Let $a, b, c, d, e \in \mathcal{S}$ denote five distinct planes. Then the diagonal-ratio factorizes into cross-ratios*

$$\operatorname{dr}(a, b, c, d, e) = \operatorname{cr}(a, b, e, d) \cdot \operatorname{cr}(a, e, c, d) \cdot \operatorname{cr}(a, d, b, c). \tag{14}$$

Proof. Following the proof of the analogous Lemma 7 for Möbius geometry, we express the diagonal-ratio in terms of tangent vectors:

$$\begin{aligned} \operatorname{dr}(a, b, c, d, e) = & \big((a-b)(b-e)^{-1}(e-a)\big)\big((a-d)^{-1}(d-e)(e-a)^{-1}\big) \\ & \big((a-e)(e-c)^{-1}(c-a)\big)\big((a-c)^{-1}(c-d)(d-a)^{-1}\big) \\ & \big((a-d)(d-b)^{-1}(b-a)\big)\big((a-b)^{-1}(b-c)(c-a)^{-1}\big) \\ = & \frac{-1}{|t_{eb}|^2}([0, t_{eb}] + \epsilon\lambda_{be})([0, t_{ed}] + \epsilon\lambda_{ed}) \\ & \frac{-1}{|t_{ce}|^2}([0, t_{ce}] + \epsilon\lambda_{ec})([0, t_{dc}] + \epsilon\lambda_{dc}) \\ & \frac{-1}{|t_{bd}|^2}([0, t_{bd}] + \epsilon\lambda_{db})([0, t_{cb}] + \epsilon\lambda_{cb}). \end{aligned}$$

With the representation (10) for the cross-ratio we easily find the claimed cross-ratios in the expression of the diagonal-ratio. □

Lemma 17 also implies the Laguerre analogue of Corollary 9.

Corollary 18. *Let $a, b, c, d, e \in \mathcal{S}$ be five oriented planes that touch a rotational cone. Then* $\operatorname{dr}(a, b, c, d, e) \in \mathbb{R}$.

Proof. If all the planes a, b, c, d, e touch a rotational cone, then each four of them also touch the same rotational cone. Therefore all three cross-ratios in Lemma 17 are real, hence $\operatorname{dr}(a, b, c, d, e)$ is real. □

Lemma 19. *Let $a, b, c, d, e \in \mathcal{S}$ and let q be the quaternion-part of* $\operatorname{dr}(a, b, c, d, e)$. *Then* $\operatorname{Im} q$ *represents the normal vector n_A of plane A.*

Proof. By partially expanding the last term of the proof of Lemma 17 we get

$$\begin{aligned}\mathrm{dr}(a,b,c,d,e) = \frac{-1}{|t_{eb}|^2|t_{ce}|^2|t_{bd}|^2}\Big(&\big([-\langle t_{eb},t_{ed}\rangle, t_{eb}\times t_{ed}] + \epsilon[0,\lambda_{ed}t_{eb}+\lambda_{be}t_{ed}]\big)\\ &\big([-\langle t_{ce},t_{dc}\rangle, t_{ce}\times t_{dc}] + \epsilon[0,\lambda_{dc}t_{ce}+\lambda_{ec}t_{dc}]\big)\\ &\big([-\langle t_{bd},t_{cb}\rangle, t_{bd}\times t_{cb}] + \epsilon[0,\lambda_{cb}t_{bd}+\lambda_{db}t_{cb}]\big)\Big).\end{aligned}$$

By the same argument as in the proof of Lemma 14, the dual quaternions in the three parentheses all have quaternion-parts whose vector-parts are parallel to n_A. Since the product of dual quaternions with linearly dependent vector-parts yields again a dual quaternion with a vector-part parallel to the ones of its factors, $\mathrm{dr}(a,b,c,d,e)$ must also have a vector-part parallel to n_A. □

Lemma 20. *Let $a,b,c,d,e\in\mathcal{S}$ and let q_1 and q_2 be the quaternion-parts of* $\mathrm{dr}(a,b,c,d,e)$ *and* $\mathrm{dr}(a,e,d,c,b)$*, respectively. Then $q_1=q_2$.*

Proof. The proof of Lemma 17 yields the representation of $\mathrm{dr}(a,b,c,d,e)$ in terms of tangent vectors in n_A (as point of the unit sphere). We set $\alpha := -1/(|t_{be}|^2|t_{db}|^2|t_{ec}|^2)$ and use the notation $\mathcal{O}(\epsilon)=\epsilon\cdot *\in\epsilon\mathbb{H}$ for a purely dual quaternion, i.e., with vanishing quaternion-part. We get

$$\begin{aligned}\mathrm{dr}(a,b,c,d,e) &= \alpha(t_{eb}+\epsilon\lambda_{be})(t_{ed}+\epsilon\lambda_{ed})(t_{ce}+\epsilon\lambda_{ec})(t_{dc}+\epsilon\lambda_{dc})(t_{bd}+\epsilon\lambda_{db})(t_{cb}+\epsilon\lambda_{cb})\\ &= \alpha(t_{eb}+\epsilon\lambda_{be})([-\langle t_{ed},t_{ce}\rangle, t_{ed}\times t_{ce}]+\mathcal{O}(\epsilon))(t_{dc}+\epsilon\lambda_{dc})([-\langle t_{bd},t_{cb}\rangle, t_{bd}\times t_{cb}]+\mathcal{O}(\epsilon)).\end{aligned}$$

Setting $r_1 := -\langle t_{bd},t_{cb}\rangle$, $r_2 := -\langle t_{ed},t_{ce}\rangle$, $n_1 := -t_{bd}\times t_{cb}$, $n_2 := t_{ed}\times t_{ce}$, and $t+\epsilon\lambda := t_{dc}+\epsilon\lambda_{dc}$ we obtain for the above expression

$$\begin{aligned}&\alpha(t_{eb}+\epsilon\lambda_{be})([r_2,n_2]+\mathcal{O}(\epsilon))\,(t+\epsilon\lambda)([r_1,-n_1]+\mathcal{O}(\epsilon))(t+\epsilon\lambda)\,(t+\epsilon\lambda)^{-1}\\ &\overset{(7)}{=} -\alpha(t_{eb}+\epsilon\lambda_{be})([r_2,n_2]+\mathcal{O}(\epsilon))\,|t|^2([r_1,n_1]+\mathcal{O}(\epsilon))\,(t+\epsilon\lambda)^{-1}.\end{aligned}$$

By substituting indices we also get the corresponding expression for

$$\mathrm{dr}(a,e,d,c,b) = -\alpha(t_{be}+\epsilon\lambda_{eb})([r_1,n_1]+\mathcal{O}(\epsilon))\,|t|^2([r_2,n_2]+\mathcal{O}(\epsilon))\,(-1)(t+\epsilon\lambda)^{-1}$$

and therefore

$$\begin{aligned}&\mathrm{dr}(a,b,c,d,e)-\mathrm{dr}(a,e,d,c,b) = -\alpha(t_{eb}+\epsilon\lambda_{be})|t|^2\\ &\cdot\big[([r_2,n_2]+\mathcal{O}(\epsilon))([r_1,n_1]+\mathcal{O}(\epsilon)) - ([r_1,n_1]+\mathcal{O}(\epsilon))([r_2,n_2]+\mathcal{O}(\epsilon))\big]\cdot(t+\epsilon\lambda)^{-1}.\end{aligned}$$

Since n_1 and n_2 are both parallel to n_A, the quaternion-parts of the dual quaternions

$$([r_2,n_2]+\mathcal{O}(\epsilon))([r_1,n_1]+\mathcal{O}(\epsilon)) \quad\text{and}\quad ([r_1,n_1]+\mathcal{O}(\epsilon))([r_2,n_2]+\mathcal{O}(\epsilon))$$

are equal. Therefore

$$\mathrm{dr}(a,b,c,d,e)-\mathrm{dr}(a,e,d,c,b) = -\alpha(t_{eb}+\epsilon\lambda_{be})|t|^2(0+\mathcal{O}(\epsilon))(t+\epsilon\lambda)^{-1} = \mathcal{O}(\epsilon),$$

i.e., the quaternion-parts of $\mathrm{dr}(a,b,c,d,e)$ and $\mathrm{dr}(a,e,d,c,b)$ are equal. □

Theorem 21. *Let A,B,C,D,E be five oriented planes in $\mathbb{R}^3$ which touch a common oriented sphere. Then the corresponding dual quaternions $a,b,c,d,e\in\mathcal{S}$ satisfy*

$$\mathrm{dr}(a,b,c,d,e)=\mathrm{dr}(a,e,d,c,b).$$

Proof. Lemma 20 implies that the quaternion-parts of $\mathrm{dr}(a,b,c,d,e)$ and $\mathrm{dr}(a,e,d,c,b)$ are equal. Therefore, it remains to prove that here their dual parts coincide as well.

Let us first bring the cross-ratio into a form which we can better deal with. Using Equation (10), the first cross-ratio in Equation (14) equals

$$\mathrm{cr}(a,b,e,d) = \frac{1}{|t_{eb}|^2}\big([-\langle t_{eb}, t_{de}\rangle, t_{eb}\times t_{de}] + \epsilon[0, \lambda_{de}t_{eb} + \lambda_{be}t_{de}]\big).$$

We set (similar to the proof of Theorem 15 without the denominators):

$$r_c^e := -\langle t_{eb}, t_{de}\rangle, \qquad n_c^e := t_{eb}\times t_{de}, \qquad v_c^e := (\lambda_{de}t_{eb} + \lambda_{be}t_{de}).$$

Lemma 14 and its proof imply that n_c^e is parallel to n_A which yields

$$n_c^e = \mu_c^e n_A$$

for some $\mu_c^e \in \mathbb{R}\setminus\{0\}$. Let us denote by m the center of the common sphere which is in contact with all planes and that exists by assumption. Then for $m_1 = m_2 = m$, Equation (12) implies

$$v_c^e = \langle m_2, t_{de}\rangle t_{eb} + \langle m_1, t_{be}\rangle t_{de} = \langle m, t_{de}\rangle t_{eb} - \langle m, t_{eb}\rangle t_{de} = m\times(t_{eb}\times t_{de}) = m\times n_c^e = \mu_c^e v, \tag{15}$$

where we define $v := m\times n_A$. Note that $n_c^e \perp v_c^e$ and also $n_A \perp v$. The cross-ratio can now be written as

$$\mathrm{cr}(a,b,e,d) = \frac{1}{|t_{eb}|^2}\big([r_c^e, \mu_c^e n_A] + \epsilon\mu_c^e v\big).$$

By setting $\alpha := (|t_{eb}|^2|t_{ce}|^2|t_{bd}|^2)$ the diagonal-ratio becomes (after carefully expanding)

$$\begin{aligned}\alpha\,\mathrm{dr}(a,b,c,d,e) &= \alpha\,\mathrm{cr}(a,b,e,d)\cdot\mathrm{cr}(a,e,c,d)\cdot\mathrm{cr}(a,d,b,c)\\ &= \big([r_c^e,\mu_c^e n_A] + \epsilon\mu_c^e v\big)\cdot\big([r_b^c,\mu_b^c n_A] + \epsilon\mu_b^c v\big)\cdot\big([r_e^b,\mu_e^b n_A] + \epsilon\mu_e^b v\big)\\ &= \ldots = * + \epsilon\big(r_c^e r_b^c\mu_e^b + r_c^e r_e^b\mu_b^c + r_b^c r_e^b\mu_c^e - \mu_c^e\mu_b^c\mu_e^b\big)v.\end{aligned}$$

From $n_c^e = \mu_c^e n_A$ we obtain (the symbol $\measuredangle(\cdot,\cdot)$ denotes the non-oriented angle)

$$\mu_c^e = \langle n_c^e, n_A\rangle = \langle t_{eb}\times t_{de}, n_A\rangle = \langle |t_{eb}||t_{de}|\sin\measuredangle(t_{be},t_{de})\frac{n_c^e}{|n_c^e|}, n_A\rangle = |t_{eb}||t_{de}|\sin\varphi_c^e,$$

where $\varphi_c^e := \langle \frac{n_c^e}{|n_c^e|}, n_A\rangle\measuredangle(t_{be},t_{de})$ is a signed angle. On the other hand the angle satisfies

$$\cos\varphi_c^e|t_{eb}||t_{de}| = \langle t_{eb}, t_{de}\rangle = -r_c^e, \tag{16}$$

which implies

$$\mu_c^e = -r_c^e\tan\varphi_c^e, \quad \text{and analogously} \quad \mu_b^c = -r_b^c\tan\varphi_b^c, \quad \text{and} \quad \mu_e^b = -r_e^b\tan\varphi_e^b.$$

Therefore,

$$\alpha\,\mathrm{dr}(a,b,c,d,e) = * - \epsilon r_c^e r_b^c r_e^b\big(\tan\varphi_e^b + \tan\varphi_b^c + \tan\varphi_c^e - \tan\varphi_c^e\tan\varphi_b^c\tan\varphi_e^b\big)v.$$

Lemma 23 (from the appendix) implies

$$\begin{aligned}\alpha\,\mathrm{dr}(a,b,c,d,e) &= * - \epsilon r_c^e r_b^c r_e^b\frac{\sin(\varphi_e^b+\varphi_b^c+\varphi_c^e)}{\cos\varphi_e^b\cos\varphi_b^c\cos\varphi_c^e}v\\ &\overset{(16)}{=} * + \epsilon|t_{eb}||t_{de}||t_{ce}||t_{dc}||t_{bd}||t_{cb}|\sin(\varphi_e^b+\varphi_b^c+\varphi_c^e)v.\end{aligned}$$

Analogously, the other diagonal-ratio takes the form

$$\alpha\,\mathrm{dr}(a,e,d,c,b) = * + \epsilon|t_{be}||t_{cb}||t_{db}||t_{cd}||t_{ec}||t_{de}|\sin(\varphi_b^e+\varphi_e^d+\varphi_d^b)v.$$

Consequently,

$$\mathrm{dr}(a,b,c,d,e) = \mathrm{dr}(a,e,d,c,b) \iff \sin(\varphi_e^b+\varphi_b^c+\varphi_c^e) = \sin(\varphi_b^e+\varphi_e^d+\varphi_d^b). \tag{17}$$

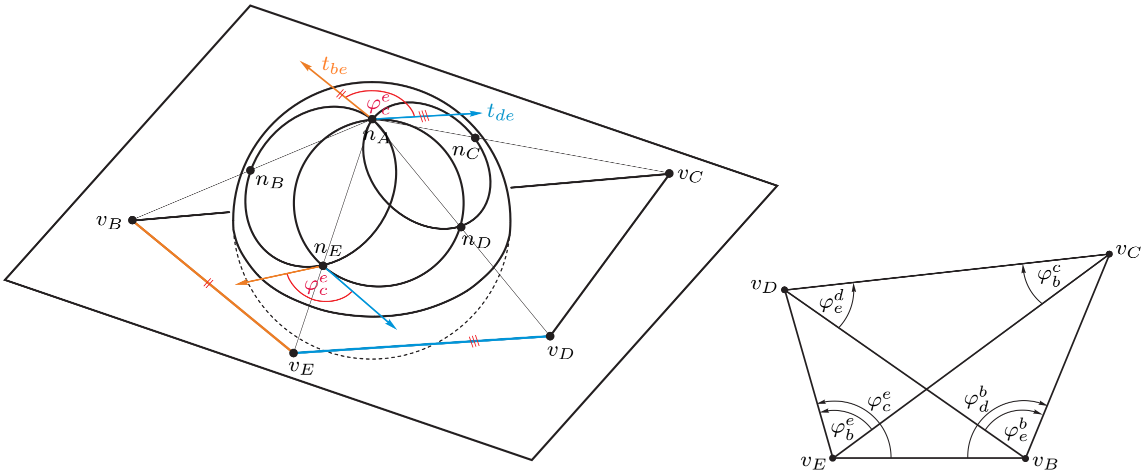

FIGURE 8. *Left:* Stereographic projection of the circumcircles of n_A, n_B, n_E, n_A, n_E, n_D, n_A, n_D, n_C. The angle φ_c^e between t_{be} and t_{de} is measured in the projection between the edges $v_B - v_E$ and $v_D - v_E$. *Right:* A quadrilateral with diagonals and oriented angles which fulfill the angle condition stated in Lemma 24.

The angles on the right-hand side are oriented angles of tangent vectors $t_{eb}, \ldots$ of circumcircles through $n_A, n_E, n_B, \ldots$ on the unit sphere (see Figure 8 left). A stereographic projection with n_A as center maps the circles to the edges of a polygon with vertices v_B, v_C, v_D, v_E. Since angles are preserved under steregraphic projection we have $\varphi_c^e = \angle(v_B - v_E, v_D - v_E)$ measured with orientation.

Now equality on the right-hand side of (17) follows from Lemma 24 in the appendix which concludes the proof. □

Corollary 22. *Let A, B, C, D be four planes in oriented contact with a unique sphere with center m. Then the dual part of $\operatorname{cr}(a, b, c, d)$ is parallel to the cross product*

$$m \times n_A,$$

where n_A is the normal vector of plane A.

Proof. Equation (15) implies that the dual part is parallel to $m \times (t_{cb} \times t_{dc})$ and Lemma 14 and its proof imply that $t_{cb} \times t_{dc}$ is parallel to n_A. □

APPENDIX

Lemma 23. *For three angles α, β, γ we have the identity*

$$\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma = \frac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}.$$

Proof. Using standard trigonometric identities yields

$$\begin{aligned}
&(\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma) \cdot (\cos\alpha\cos\beta\cos\gamma) \\
=\ & \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\alpha\cos\gamma + \sin\gamma\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma \\
=\ & \sin\alpha(\cos\beta\cos\gamma - \sin\beta\sin\gamma) + \cos\alpha(\sin\beta\cos\gamma + \cos\beta\sin\gamma) \\
=\ & \sin\alpha\cos(\beta+\gamma) + \cos\alpha\sin(\beta+\gamma) = \sin(\alpha+\beta+\gamma),
\end{aligned}$$

which concludes the proof. □

Lemma 24. *Let $v_B, v_C, v_D, v_E \in \mathbb{C}$ be a quadrilateral and let its oriented angles be denoted by*

$$\begin{aligned} \varphi_c^e &= \angle(v_B - v_E, v_D - v_E) & \varphi_d^b &= \angle(v_E - v_B, v_C - v_B) \\ \varphi_b^c &= \angle(v_E - v_C, v_D - v_C) & \varphi_e^d &= \angle(v_B - v_D, v_C - v_D) \\ \varphi_e^b &= \angle(v_D - v_B, v_C - v_B) & \varphi_b^e &= \angle(v_C - v_E, v_D - v_E) \end{aligned}$$

(see Figure 8 right). Then

$$\sin(\varphi_c^e + \varphi_b^c + \varphi_e^b) = \sin(\varphi_d^b + \varphi_e^d + \varphi_b^e).$$

Proof. For an oriented angle $\angle(a, b)$ between two complex numbers $a, b \in \mathbb{C}$ the exponential $e^{i\angle(a,b)}$ is equal to $\frac{b}{|b|} / \frac{a}{|a|}$. Consequently,

$$\frac{e^{i(\varphi_c^e + \varphi_b^c + \varphi_e^b)}}{e^{i(\varphi_d^b + \varphi_e^d + \varphi_b^e)}} = \frac{\frac{v_D - v_E}{v_B - v_E} \frac{v_D - v_C}{v_E - v_C} \frac{v_C - v_B}{v_D - v_B}}{\frac{v_C - v_B}{v_E - v_B} \frac{v_C - v_D}{v_B - v_D} \frac{v_D - v_E}{v_C - v_E}} = 1,$$

which concludes the proof. □

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References

[1] Alexander I. Bobenko and Ulrich Pinkall. Discrete isothermic surfaces. *J. Reine Angew. Math.*, 475:187–208, 1996.

[2] Oene Bottema and Bernard Roth. *Theoretical kinematics*, volume 24 of *North-Holland Series in Applied Mathematics and Mechanics*. North-Holland Publishing Co., Amsterdam-New York, 1979.

[3] Jan Cieśliński. The cross ratio and Clifford algebras. *Adv. Appl. Clifford Algebras*, 7(2):133–139, 1997.

[4] Daoming Liu, Davide Pellis, Yu-Chou Chiang, Florian Rist, Johannes Wallner, and Helmut Pottmann. Deployable strip structures. *ACM Trans. Graphics*, 42(4):103:1–16, 2023. Proc. SIGGRAPH.

[5] Christian Müller and Amir Vaxman. Discrete curvature and torsion from cross-ratios. *Ann. Mat. Pura Appl. (4)*, 200(5):1935–1960, 2021.

[6] Davide Pellis, Hui Wang, Florian Rist, Martin Kilian, Helmut Pottmann, and Christian Müller. Principal symmetric meshes. *ACM Trans. Graphics*, 39(4):127:1–17, 2020. Proc. SIGGRAPH.

[7] Eike Schling, Martin Kilian, Hui Wang, Denis Schikore, and Helmut Pottmann. Design and construction of curved support structures with repetitive parameters. In L. Hesselgren et al., editor, *Advances in Architectural Geometry*, pages 140–165. Klein Publishing Ltd, 2018.

[8] Eduard Study. *Geometrie der Dynamen: Die Zusammensetzung von Kräften und verwandte Gegenstände der Geometrie*. B.G. Teubner, 1903.

[9] Amir Vaxman, Christian Müller, and Ofir Weber. Canonical Möbius subdivision. *ACM Trans. Graph.*, 37(6):227:1–15, 2018.

[10] Isaak M. Yaglom. *Complex numbers in geometry*. Academic Press, New York-London, 1968. Translated from the Russian by Eric J. F. Primrose.