Surfaces of constant principal-curvatures ratio in isotropic geometry

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Abstract

We study surfaces with a constant ratio of principal curvatures in Euclidean and simply isotropic geometries and characterize rotational, channel, ruled, helical, and translational surfaces of this kind under some technical restrictions (the latter two cases only in isotropic geometry). We use the interlacing of various methods of differential geometry, including line geometry and Lie sphere geometry, ordinary differential equations, and elementary algebraic geometry.

Keywords: Isotropic geometry, constant ratio of principal curvatures, minimal surfaces, Weingarten surfaces.

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Contents

1 Introduction

We study surfaces with a constant ratio of principal curvatures in Euclidean and simply isotropic geometries and characterize rotational, channel, ruled, helical, and translational surfaces of this kind under some technical restrictions (the latter two cases only in isotropic geometry).

Surfaces with a constant ratio of principal curvatures, or briefly *CRPC surfaces*, generalize minimal surfaces while keeping invariance under similarities. However, they are significantly harder to construct than minimal surfaces. CRPC surfaces are characterized geometrically as surfaces having a constant angle between characteristic curves (asymptotic curves in the case of negative Gaussian curvature; conjugate and principal symmetric curves in case of positive Gaussian curvature).

Recent interest in CRPC surfaces has its origin in architecture, in particular in the aim of building geometrically complex shapes from simple elements. A remarkable class of such shapes is given by the asymptotic gridshells of E. Schling [1, 2]. See Fig. 1. They are formed by bending originally flat straight lamellas of bendable material (metal, timber) and arranging them in a quadrilateral

Fig. 1: An asymptotic gridshell (top) [1]. Straight lamellas (middle) follow asymptotic curves of the reference surface and intersect under a right angle. This simplifies the manufacturing process as all steel joints (bottom) are identical but forces the reference surface to be a Euclidean minimal surface. If the intersection angle is constant and not right, then the joints are still identical, and we get more general surfaces: ones with a constant ratio of principal curvatures.

structure so that all strips are orthogonal to some reference surface S. This requires the strips to follow asymptotic curves on S. If, in addition, one aims at congruent nodes to further simplify fabrication, one arrives at surfaces S on which asymptotic directions form a constant angle, i.e., at negatively curved CRPC surfaces. Even positively curved CRPC surfaces and other Weingarten surfaces are of interest in architecture, since they only have a one-parameter family of curvature elements, which simplifies surface paneling of double-curved architectural skins through mold re-use [3].

A classical general approach to complicated problems in Euclidean geometry is to start with their simpler analogs in so-called simply isotropic geometry. The isotropic analogs give a lot of geometric insight and also provide an initial guess for numerical optimization. This approach has been (implicitly) used since as early as the work [4] by Müntz from 1911, who solved the Plateau problem for Euclidean minimal surfaces in a quite general setup by deformation of graphs of harmonic functions. Such graphs are minimal surfaces in simply isotropic geometry; thus in this case the optimization led to the whole existence proof.

Simply isotropic geometry, also called just *isotropic geometry* in the literature and the rest of this paper, has been studied extensively by K. Strubecker (see, e.g., [5–7]) and is treated in the monograph by H. Sachs [8]. It is based on the group of affine transformations which preserve the *isotropic semi-norm* $\|(x, y, z)\|_i := \sqrt{x^2 + y^2}$ in space with the coordinates x, y, z. It can also be seen as relative differential geometry with respect to the *unit isotropic sphere* (paraboloid of revolution) $2z = x^2 + y^2$ (see [9–11]). Isotropic geometry is simpler but has much in common with Euclidean and other Cayley–Klein geometries.

The isotropic geometry of surfaces appears also in structural design and statics (see, e.g., [12]), due to the close relation between the stresses in a planar body and the isotropic curvatures of the associated Airy stress surface [13]. CRPC surfaces in isotropic space represent planar stress states with a constant ratio of principal stresses.

In our arguments, we use the interlacing of various methods of differential geometry, ordinary differential equations, and elementary algebraic geometry (the latter — in the classification of the translational CRPC surfaces, which we consider as our main contribution; see Section 7).

1.1 Previous work

Euclidean Geometry. Only a few explicit examples of CRPC surfaces, not being minimal surfaces, have been known before. The explicit parameterizations were only available for rotational and helical CRPC surfaces [14–20].

CRPC surfaces are a special case of so-called Weingarten surfaces. A *Weingarten surface* is a surface with a fixed functional relation $f(H, K) = 0$ between the mean curvature H and Gaussian curvature K at each point (we assume that the zero set of the function f is an analytic curve). A surface is called *linear Weingarten* if there is a fixed linear relation between the two principal curvatures κ_1 and κ_2 at each point. Recently López and Pámpano [18] have classified all rotational linear Weingarten surfaces, which are CRPC surfaces when the intercept of the relation is zero. Moreover, it has been shown that linear Weingarten surfaces are rotational if they are foliated by a family of circles [21]. Using quite involved computations, Havlíček [22] proved that channel Weingarten surfaces must be rotational or pipe surfaces; cf. [23]. In Section 3 we give a geometric proof of this result.

In works [24–26] rotational CRPC surfaces with $K < 0$ have been characterized via isogonal asymptotic parameterizations. Yang et al. [20] have recently presented a characterization of all helical CRPC surfaces.

The most well-studied Weingarten surfaces are the ones with $K = \mathrm{const}$ or $H = \mathrm{const}$. Classification results for rotational, helical, and translational surfaces of this kind can be found in [27, Chapter 26, p. 147–158] and [28–33]. Recently Udo et al. [34] obtained explicit parametrization for channel surfaces with $K = \mathrm{const}$ in the space forms.

CRPC surfaces, via a Christoffel-type transformation of certain spherical nets, were derived in [35] with a focus on discrete models. In work [19], we can find an effective method for the computation of discrete CRPC surfaces that provides insight into the shape variety of CRPC surfaces. Since numerical optimization was involved there, one cannot derive precise mathematical conclusions, but it can be helpful for further studies.

Isotropic Geometry. As far as we know, the examples of isotropic CRPC surfaces known before were either minimal (having isotropic mean curvature $H = 0$) or paraboloids (having both $H = \mathrm{const}$ and the isotropic Gaussian curvature $K = \mathrm{const}$). However, there is a variety of related works regarding the conditions $H = \mathrm{const}$ or $K = \mathrm{const}$ separately. Surfaces with $K = \mathrm{const}$ have received early attention as solutions of the Monge-Ampére equation, but only within isotropic geometry their geometric constructions, e.g., as Clifford translational surfaces, are elegant and simple [6]. Invariant surfaces with $H = \mathrm{const}$ or $K = \mathrm{const}$, including the parabolic rotational surfaces, were studied in detail by da Silva [36]. An exact representation of several types of ruled surfaces with $H = \mathrm{const}$ or $K = \mathrm{const}$ can be found in [37, 38]. All helical surfaces with $H = \mathrm{const}$ or $K = \mathrm{const}$ were classified in [39, 40]. Translational surfaces with $H = \mathrm{const}$ or $K = \mathrm{const}$ were classified in [41], (generalizing the classical result for $H = 0$ from [42]) in the case when the generating curves are planar and in [43] in the case when one of the generating curves is spatial. However, the classification is still unknown when both generating curves are spatial.

2 Preliminaries

2.1 Admissible surfaces and isotropic curvatures

Recall that the *isotropic semi-norm* in space with the coordinates x, y, z is $\|(x, y, z)\|_i := \sqrt{x^2 + y^2}$. An affine transformation of $\mathbb{R}^3$ that scales the isotropic semi-norm by a constant factor has the form

$$\mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}, \quad A = \begin{pmatrix} \pm h_1 & \mp h_2 & 0 \\ h_2 & h_1 & 0 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

for some values of the parameters $\mathbf{b} \in \mathbb{R}^3$ and $h_1, h_2, c_1, c_2, c_3 \in \mathbb{R}$. Such transformations form the 8-parametric group $\mathcal{G}^8$ of *general isotropic similarities*.

The group of *isotropic congruences* is the 6-parametric subgroup with

$$h_1 = \cos\phi, \quad h_2 = \sin\phi, \quad c_3 = 1.$$

These transformations appear as Euclidean congruences in the projection onto the plane $z = 0$, which we call *top view*. Therefore, *isotropic distances* between points and *isotropic angles* between lines appear in the top view as Euclidean distances and angles, respectively.

Lines and planes that are parallel to the z-axis are called *isotropic* or *vertical*. They play a special role and are usually excluded as tangent spaces in differential geometry. A point of a surface is *admissible* if the tangent plane at the point is non-isotropic, and a surface is *admissible* if it has only admissible points. Hereafter by a *surface* we mean the image of a proper injective C^3 map of a closed planar domain into $\mathbb{R}^3$ with nondegenerate differential at each point, or, more generally, an embedded connected 2-dimensional C^3 submanifold of $\mathbb{R}^3$, possibly with boundary and possibly non-compact.

An admissible surface can be locally represented as the graph of a function,

$$z = f(x, y).$$

It is natural to measure the curvature of the surface in a given direction by a second-order quantity invariant under isotropic congruences and vanishing for a plane. Thus the *isotropic normal curvature* in a tangent direction $t = (t_1, t_2, t_3)$ with $\|t\|_i = t_1^2 + t_2^2 = 1$ is defined to be the second directional derivative of f,

$$\kappa_n(t) = (t_1, t_2) \cdot \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ t_2 \end{pmatrix},$$

and the *isotropic shape operator* is defined to be the Hessian $\nabla^2(f)$ of f. Its eigenvalues κ_1 and κ_2 are the *isotropic principal curvatures* and the eigenvectors are the *isotropic principal directions*. For $\kappa_1 \neq \kappa_2$, the latter are orthogonal in the top view, and thus also orthogonal in the isotropic sense.

The *isotropic mean* and *Gaussian curvatures* are defined respectively by

$$H := \frac{\kappa_1 + \kappa_2}{2} = \frac{f_{xx} + f_{yy}}{2} \qquad \text{and} \qquad K := \kappa_1\kappa_2 = f_{xx}f_{yy} - f_{xy}^2.$$

An admissible surface *has a constant ratio a of isotropic principal curvatures*, or is a *CRPC surface*, if $K \neq 0$, and $\kappa_1/\kappa_2 = a$ or $\kappa_2/\kappa_1 = a$ at each point of the surface. The latter condition is equivalent to $H^2/K = (a+1)^2/(4a)$.

In particular, for $a = -1$ we get *isotropic minimal surfaces*, characterized by the condition $H = 0$, i.e., the graphs of harmonic functions. As another example, for $a = 1$ we get a unique up to isotropic similarity CRPC surface $2z = x^2 + y^2$, also known as the *isotropic unit sphere* [7, Section 62, p. 402].

Isotropic principal curvature lines, asymptotic curves, and *isotropic characteristic curves* are defined analogously to the Euclidean case as curves tangent to corresponding directions (we exclude the points where $\kappa_1 = \kappa_2$). Recall that two directions (t_1, t_2, t_3) and (s_1, s_2, s_3) at a surface point are *conjugate* if $f_{xx}t_1s_1 + f_{xy}t_1s_2 + f_{yx}t_2s_1 + f_{yy}t_2s_2 = 0$. One can see that two directions are conjugate if one is tangent to a curve on the surface, while the other one is a ruling of the envelope of the tangent planes at points on the curve (see, e.g., [44, Section 60], [45, Sections 2-10]). In particular, the isotropic principal directions are the ones that are conjugate and orthogonal in the top view. For $K > 0$, the *isotropic characteristic directions* are the ones that are conjugate and symmetric with respect to the isotropic principal directions in the top view. For $K < 0$, they coincide with the asymptotic directions, which are the same in Euclidean and isotropic geometry.

For isotropic CRPC surfaces, the isotropic characteristic curves intersect under the constant isotropic angle γ with $\cot^2(\gamma/2) = |a|$. To see this, make the tangent plane at the intersection point horizontal by an appropriate isotropic congruence of the form $z \mapsto z+px+qy$ and apply a similar assertion in Euclidean geometry [20, Section 2.1]. All sufficiently smooth Euclidean or isotropic CRPC surfaces are analytic by the Petrowsky theorem [46, p. 3–4]. We sometimes restrict our results to the case of analytic surfaces, if this simplifies the proofs.

Example 1. For any paraboloid with a vertical axis, or, equivalently, the graph of any quadratic function $f(x, y)$ with $\det \nabla^2(f) \neq 0$, both isotropic principal curvatures are constant. Hence their ratio a is also constant. This surface can be brought to the paraboloid $z = x^2 + ay^2$ by an appropriate general isotropic similarity. (Technically, $1/a$ can also be considered as such a ratio of the same surface, but $z = x^2 + ay^2$ is isotropic similar to $z = x^2 + y^2/a$.)

The isotropic principal curvature lines of a non-rotational paraboloid $z = x^2 + ay^2$, where $a \neq 0, 1$, are parabolae (parabolic isotropic circles, to be discussed below) in the isotropic planes $x = \text{const}$ and $y = \text{const}$. The rotational paraboloid $z = x^2 + ay^2 + (a - 1)(x - x_0)^2$ is tangent to the surface $z = x^2 + ay^2$ along the isotropic principal curvature line $x = x_0$. Thus the surface is an envelope of a one-parameter family (actually, two families) of congruent rotational paraboloids with vertical axes (known as parabolic isotropic spheres).

The characteristic curves of the paraboloid $z = x^2 + ay^2$, where $a \neq 0, 1$, appear in the top view as lines parallel to $x = \pm\sqrt{|a|}y$. For a hyperbolic paraboloid ($a < 0$), these curves are the rulings. For an elliptic paraboloid ($a > 0$), they are parabolae forming a translational net on the surface.

Thus a paraboloid with a vertical axis is both a translational (see Section 7), a parabolic rotational (see Section 3), isotropic channel (see Section 4), and, for $a < 0$, a ruled surface (see Section 5).

2.2 Isotropic spheres and circles

In isotropic geometry, there are two types of spheres.

The set of all points at the same isotropic distance r from a fixed point O is called a *cylindrical isotropic sphere*. In Euclidean terms, it can be visualized as a right circular cylinder with vertical rulings. Its top view appears as a Euclidean circle with the center at the top view of O and the radius r. Any point on the axis of this cylinder can serve as the center of the same isotropic sphere.

An (inclusion-maximal) surface with both isotropic principal curvatures equal to a constant $A \neq 0$ is a *parabolic isotropic sphere*. It has the equation

$$2z = A\left(x^2 + y^2\right) + Bx + Cy + D, \qquad A \neq 0,$$

for some $B, C, D \in \mathbb{R}$. Here $1/A$ is called the *radius* of the isotropic sphere. Such isotropic spheres are paraboloids of revolution with vertical axes.

The intersection of an isotropic sphere S with a non-tangential plane P is an *isotropic circle*. The isotropic circle is *elliptic* if P is non-isotropic, *parabolic* if S is parabolic and P is isotropic, *cylindrical* if S is cylindrical and P is isotropic. The resulting isotropic circle is an ellipse whose top view is a Euclidean circle, a parabola with a vertical axis, or a pair of vertical lines, respectively.

Recall that two curves $\mathbf{x}(t)$ and $\mathbf{y}(t)$ have a *second-order contact* for $t = 0$, if $\mathbf{x}(0) = \mathbf{y}(0)$, $\mathbf{x}'(0) = \mathbf{y}'(0)$, and $\mathbf{x}''(0) = \mathbf{y}''(0)$. Two non-parameterized curves have a *second-order contact* if some of their regular parametrizations do. The *osculating isotropic circle* of a spatial curve at a non-inflection point is an isotropic circle having a second-order contact with the curve at the point.

There is an analog of Meusnier's theorem in isotropic geometry.

Theorem 2. (See [8, Theorem 9.3], [7, Section 47]) *Let an admissible surface Φ have isotropic normal curvature $\kappa_n \neq 0$ at a point $p \in \Phi$ along a surface tangent line T. Then the osculating isotropic circles of all curves on Φ that are tangent to T at p lie on the parabolic isotropic sphere of radius $1/\kappa_n$ tangent to Φ at p.*

3 Rotational surfaces

3.1 Isotropic rotational CRPC surfaces

Euclidean rotations about the z-axis are also isotropic congruences. A surface invariant under these rotations is called *isotropic rotational*, as well as the image of the surface under any isotropic congruence.

Looking for isotropic rotational CRPC surfaces, we consider the graph of a smooth function $z = h(r)$ of the *radial distance* $r := \sqrt{x^2 + y^2}$. Profiles $x/y = \text{const}$ and parallel circles $r = \text{const}$ give a principal parameterization in isotropic geometry, due to the symmetries. The isotropic profile curvature κ_2 equals the 2nd derivative $h''(r)$, and $\kappa_1 = h'(r)/r$ by Meusnier's theorem (Theorem 2). Hence $\kappa_2/\kappa_1 = a$ amounts to solutions of $ah' = rh''$. Up to isotropic similarities, this yields the profile curves

$$h(r) = \begin{cases} r^{1+a}, & \text{if } a \neq -1; \\ \log r, & \text{if } a = -1, \end{cases} \tag{1}$$

and the ones with a replaced by $1/a$. We have arrived at the following result.

Proposition 3. *(See Figure 2) An admissible isotropic rotational surface has a constant ratio $a \neq 0$ of isotropic principal curvatures if and only if it is isotropic similar to a subset of one of the surfaces*

$$z = (x^2+y^2)^{(1+a)/2} \qquad \text{or} \qquad z = (x^2+y^2)^{(1+a)/(2a)}, \qquad \text{if } a \neq -1; \qquad (2)$$
$$z = \log(x^2+y^2), \qquad \text{if } a = -1. \qquad (3)$$

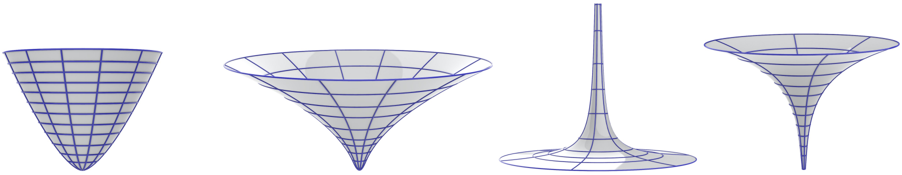

Fig. 2: Rotational isotropic CRPC surfaces (from the left to the right): the first of two surfaces (2) for $a > 0$, $0 > a > -1$, and $a < -1$ respectively; surface (3).

Let us discuss the geometry of the resulting surfaces, namely, the first of two surfaces (2) for $a \neq 0, \pm 1$. Profile curves (1) are also known as *W-curves* or $(-a)$*-catenaries* [11, Corollary 1]: they are paths of one-parameter continuous subgroups of the group of affine maps, actually, of $\mathcal{G}^8$.

The isotropic characteristic curves intersect the first isotropic principal curvature lines (parallel circles) under the constant isotropic angle $\gamma/2$ with $\cot^2(\gamma/2) = |a|$. Since isotropic angles appear as Euclidean angles in the top view, the top views of the former must be logarithmic spirals, which intersect the radial lines at angles $(\pi - \gamma)/2$. Hence, in the cylindrical coordinate system (r, ϕ, z), the isotropic characteristic curves are isotropic congruent to

$$r(\phi) = e^{\phi/\sqrt{|a|}}, \quad z(\phi) = e^{\phi(1+a)/\sqrt{|a|}}.$$

These curves are again W-curves of a one-parametric subgroup of the isotropic similarity group $\mathcal{G}^8$, and the rotational surfaces themselves are generated by the subgroup. Its elements are compositions of a rotation about the z-axis through some angle ϕ, a homothety with the center at the origin and the coefficient $e^{\phi/\sqrt{|a|}}$, and the scaling by a factor of $e^{\phi a/\sqrt{|a|}}$ in the vertical direction.

The simplest cases are $a = \pm 1$. We obtain the isotropic sphere $z = r^2$ and the logarithmoid $z = \log r$. The latter is the only isotropic minimal surface of revolution (besides planes and up to general isotropic similarities) and can be viewed as the isotropic analog of the catenoid [47, Section 4.3(a)].

Also of interest is the case $a = -1/2$, which leads to surfaces obtained by rotating the parabola $z^2 = r$ about its tangent at the vertex. Recall that in Euclidean geometry, the ratio $\kappa_2/\kappa_1 = -1/2$ also leads to parabolae as profiles, but rotated about the directrix [19]. Both surfaces are algebraic of order 4.

Clearly, we get rational algebraic surfaces for rational values of $a \neq -1$.

Yet another observation is that analytic rotational isotropic CRPC surfaces cannot intersect the rotation axis (at a nonsingular point) unless a or $1/a$ is a positive odd integer.

3.2 Euclidean rotational CRPC surfaces

It is worth comparing profile curves (1) of rotational CRPC surfaces with their analogs in Euclidean geometry ([14], [15, Ex. 3.27], [18, Eq. (3.2)], [19, Eq. (7)]):

$$h(r) = \int \frac{r^a\,dr}{\sqrt{1-r^{2a}}} = \frac{r^{1+a}}{1+a}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}+\frac{1}{2a};\ \frac{3}{2}+\frac{1}{2a};\ r^{2a}\right), \quad a \neq -1, -\frac{1}{3}, \dots \tag{4}$$

Here ${}_2F_1(\alpha, \beta;\ \gamma;\ z)$ is the Gauss hypergeometric function (see e.g. [48, Ch. V, Section 7] for a definition), $r^a \leq 1$, and $1/a$ is not a negative odd integer. The latter equality is checked in [49, Section 3].

3.3 Parabolic rotational CRPC surfaces

In isotropic geometry, there is a second type of rotations, so-called *parabolic rotations*, given by

$$\begin{aligned} x' &= x + t, \\ y' &= y, \\ z' &= t^2/2 + (x + by)t + z, \end{aligned}$$

for some parameters b, t [8, Eq. (2.14)]. A surface invariant under these transformations for b fixed and t running through $\mathbb{R}$ is called *parabolic rotational*, as well as the image of the surface under any isotropic congruence.

It is not hard to find all parabolic rotational CRPC surfaces. Indeed, let the graph of a smooth function $z = z(x, y)$ be invariant under the parabolic rotations. The section $x = 0$ is the graph of the function $h(y) := z(0, y)$. Applying the parabolic rotation with $t = x$ to the latter curve, we get the identity $z(x, y) = x^2/2 + bxy + h(y)$. Then the isotropic Gaussian and mean curvatures are $K = h'' - b^2$ and $H = (h'' + 1)/2$. Then the equation $H^2/K = (a+1)^2/(4a)$ is equivalent to $a(h'' + 1)^2 = (a + 1)^2(h'' - b^2)$. Hence $h'' = \text{const}$ and $z(x, y)$ is a quadratic function. By Example 1, we arrive at the following proposition.

Proposition 4. *An admissible parabolic rotational surface has a constant ratio $a \neq 0$ of isotropic principal curvatures if and only if it is isotropic similar to a subset of the paraboloid $z = x^2 + ay^2$.*

An analogous result for constant mean curvature surfaces was obtained in [11, Proposition 3]).

In this case, both isotropic principal curvatures are constant. In the next section, we show that this is the only surface with this property (Theorem 8).

4 Channel surfaces

Now we turn to channel surfaces and show that all channel CRPC surfaces are rotational (or parabolic rotational). Thus we will not encounter new surfaces.

4.1 Euclidean channel CRPC surfaces

A *channel surface* C is defined as the *envelope* of a smooth one-parameter family of spheres $S(t)$, i.e., the surface C that is tangent to each sphere $S(t)$ along a single closed curve $c(t)$ so that the curves $c(t)$ cover C. The curves $c(t)$ are called *characteristics* (not to be confused with the *characteristic curves* discussed in Section 2). Each characteristic $c(t)$ is a circle which is a principal curvature line on C (see Lemma 6). The locus of sphere centers $s(t)$ is referred to as a *spine curve* and their radii $r(t)$ constitute the *radius function*. Special cases of channel surfaces are *pipe surfaces*, being envelopes of congruent spheres.

For simplicity, we restrict ourselves to analytic surfaces. (Recall that all sufficiently smooth Euclidean or isotropic CRPC surfaces are analytic by the Petrowsky theorem [46, p. 3–4].) If C is analytic (and has no umbilic points), then the family $S(t)$ is also analytic up to a change of the parameter t (because the principal directions, hence the principal curvature lines $c(t)$, hence the spheres $S(t)$ analytically depend on a point of C). Thus by an *analytic channel surface* we mean an analytic surface which is the envelope of a one-parameter analytic family of spheres $S(t)$, i.e, a family such that both $s(t)$ and $r(t)$ are real-analytic functions.

We would like to give a short proof of the following result by Havlíček [22].

Theorem 5. *An analytic channel Weingarten surface is a rotational or pipe surface. In particular, if an analytic channel surface has a constant ratio of principal curvatures, then it is rotational.*

We start by recalling a well-known proof of the basic properties of characteristics; this will help us in establishing their isotropic analogs.

Lemma 6. *Under the notation at the beginning of Section 4, $c(t)$ is a principal curvature line on C, and the principal curvature along $c(t)$ is $1/r(t)$. If the family $S(t)$ is analytic then $c(t)$ is a circle with the axis parallel to $s'(t)$. If $r'(t) \neq 0$ then $s'(t) \neq 0$.*

Proof Since $S(t)$ and C are tangent along $c(t)$, and $c(t)$ is a principal curvature line of $S(t)$, it is a principal curvature line of C as well by Joachimsthal theorem.

Let us compute the principal curvature κ_1 along $c(t)$. Let p be a point on $c(t)$ and T be the tangent line to C through p. Then by Meusnier's theorem, the osculating circle of $c(t)$ lies on the sphere of radius $1/\kappa_1$ that is tangent to C at p. On the other hand, $c(t)$, hence its osculating circle, lies on the sphere $S(t)$ of radius $r(t)$ that is also tangent to C at p. This implies that $\kappa_1 = 1/r(t)$.

Distinct characteristics $c(t_1)$ and $c(t_2)$ cannot have two common points or be tangent to each other; otherwise $S(t_1)$ and $S(t_2)$ coincide and would not be tangent to C along a single closed curve. For t_2 close to t_1, the curves $c(t_1)$ and $c(t_2)$ cannot have a unique non-tangential intersection point because a neighborhood of $c(t_1)$ is tangent to $S(t_1)$ along $c(t_1)$ and hence is orientable. By continuity, for each $t_2 \neq t_1$ the curves $c(t_1)$ and $c(t_2)$ are either disjoint or coincide.

Let us compute $c(t)$. The sphere $S(t)$ has the equation $(\mathbf{x} - s(t))^2 = r(t)^2$, hence $c(t)$ is contained in the intersection of $S(t)$ with the set $(\mathbf{x} - s(t))s'(t) = r'(t)r(t)$; cf. [50, Section 5.13]. If $s'(t) \neq 0$, the latter is a plane orthogonal to $s'(t)$, hence $c(t)$ is a circle with the axis parallel to $s'(t)$ because $c(t)$ is a closed curve by the definition of a channel surface. If $s'(t) = 0$, then $c(t)$ is still a circle as a closed curve containing the limit of circles $c(t_n)$, where $t_n \to t$ and $s'(t_n) \neq 0$ (the limit exists by analyticity and does not degenerate to a point because a family of pairwise disjoint circles disjoint from a curve on a surface cannot shrink to a point on the curve).

Finally, if $r'(t) \neq 0$ then $s'(t) \neq 0$ because $c(t) \neq \emptyset$. $\square$

We now prove the theorem modulo a technical lemma and then the lemma.

Proof of Theorem 5 Consider an analytic channel Weingarten surface C and a sphere $S = S(t)$ of the enveloping family. They are tangent along the characteristic $c = c(t)$. By Lemma 6 the principal curvature along the circle c, say $\kappa_1 = \kappa_1(t)$, is constant (it is equal to the inverse of the radius of $S(t)$). If the Weingarten relation is not $\kappa_1 = \text{const}$ then the other principal curvature $\kappa_2 = \kappa_2(t)$ has to be constant along c.

First, let us consider the *general-position case*: $\kappa_1(t), \kappa_2(t), \kappa_1'(t), \kappa_2'(t) \neq 0$ and $\kappa_1(t) \neq \kappa_2(t)$ for all t. At each point of c, draw an osculating circle of the section of C by the plane orthogonal to c. Since all such circles have curvature κ_2 and are tangent to the sphere S, they are obtained by rotations of one circle about the axis of c and form a rotational surface D. By construction, C and D have a second-order contact along c. Then Lemma 7 below asserts that at least in the general-position case, the spine curve $s(t)$ of C has a second-order contact with the spine curve of D, i.e., the axis of $c(t)$. Since $s(t)$ has a second-order contact with a straight line for all t, it is a straight line and C is a rotational surface.

Next, let us reduce the theorem to the above general-position case.

If $\kappa_1(t) \equiv \kappa_2(t)$ as functions in t, then C is a subset of a sphere because $\kappa_1(t) \neq 0$.

If $\kappa_1(t) = \text{const}$, then C is a pipe surface because $1/\kappa_1(t)$ is the radius of $S(t)$.

If $\kappa_2(t) = \text{const}$, then C is a rotational surface. Indeed, consider a principal curvature line orthogonal to $c(t)$ and let $p(t)$ be its intersection point with $c(t)$. Let $n(t)$ be the surface unit normal at $p(t)$. Then $n'(t) = \kappa_2(t)p'(t)$. Hence the curvature center $p(t) - n(t)/\kappa_2(t) = \text{const}$ (or $n(t) = \text{const}$ for $\kappa_2(t) = 0$). Thus the principal curvature line lies on the sphere of radius $1/\kappa_2(t)$ that is tangent to the surface (or on a plane for $\kappa_2(t) = 0$). For the other principal curvature lines orthogonal to $c(t)$, such spheres (or planes) are obtained by rotations about the axis of the circle $c(t)$. Then C is a subset of the envelope of those spheres (or planes) and hence it is rotational.

Otherwise, restrict the range of t to an interval where the above general-position assumptions are satisfied. The envelope of the resulting sub-family $S(t)$ is rotational by the above general-position case. Then the whole C is rotational by the analyticity.

Finally, for a CRPC surface, the condition $\kappa_1(t) = \text{const}$ implies $\kappa_2(t) = \text{const}$ and yields again to a rotational surface. $\square$

The deep idea beyond this proof is that a channel surface C has a second-order contact with the *osculating Dupin cyclide* D [51] along a characteristic, and the spine curves of C and D also have a second-order contact. Here D is the limit of the envelope of all spheres tangent to three spheres $S(t_1)$, $S(t_2)$, $S(t_3)$ for t_1, t_2, t_3 tending to t. The spine curves of C and the envelope have 3 colliding common points $s(t_1), s(t_2), s(t_3)$, leading to a second-order contact. For a Weingarten surface C, the osculating Dupin cyclides D turn out to be rotational, with straight spine curves, hence the same must be true for C itself.

Although this construction gives geometric insight, passing to the limit $t_1, t_2, t_3 \to t$ rigorously is a bit technical. Thus we complete the proof by a different argument relying on additional curvature assumptions.

Lemma 7. *Under the notation of the proof of Theorem 5 and the assumptions* $\kappa_1(t), \kappa_2(t), \kappa_1'(t), \kappa_2'(t) \neq 0$ *and* $\kappa_1(t) \neq \kappa_2(t)$ *for all* t*, the spine curve* $s(t)$ *has a second-order contact with the axis of the circle* $c(t)$*.*

Proof In what follows we fix a value of t, say, $t = 0$, and assume that t is sufficiently close to this value. It is also convenient to assume that $\kappa_2'(0)(\kappa_1(0) - \kappa_2(0)) > 0$ and $t > 0$. Since $\kappa_1'(0) \neq 0$, by Lemma 6 it follows that $s'(0)$ is nonzero and parallel to the axis of $c(0)$. It remains to prove that the distance from $s(t)$ to the axis is $O(t^3)$.

We do it by reducing to a planar problem; see Fig. 3 to the left. Take a plane P passing through $s(0)$ and parallel to $s'(0)$ and $s''(0)$ (an "osculating plane" of $s(t)$). It contains the axis of $c(0)$. The section of $c(t)$ by the plane P consists of two points $c_1(t)$ and $c_2(t)$. Let c_1 and c_2 be the two curves formed by these points. The section of C by P coincides with c_1 and c_2 because the characteristics cover the surface. The section of $S(t)$ is a circle $S_1(t)$ tangent to c_1 and c_2 at $c_1(t)$ and $c_2(t)$. Let $D_1(t)$ and $D_2(t)$ be the osculating circles of c_1 and c_2 at $c_1(t)$ and $c_2(t)$. Orient each circle $S_1(t)$ counterclockwise and fix the orientations of $D_1(t)$ and $D_2(t)$ such that the contacts are oriented. Let $s_1(t)$ be the center of $S_1(t)$. Since $P \parallel s'(0), s''(0)$, it follows that the distance between $s_1(t)$ and $s(t)$ is $O(t^3)$. Thus it suffices to prove that the distance from $s_1(t)$ to the bisector of $c_1(0)c_2(0)$ is $O(t^3)$.

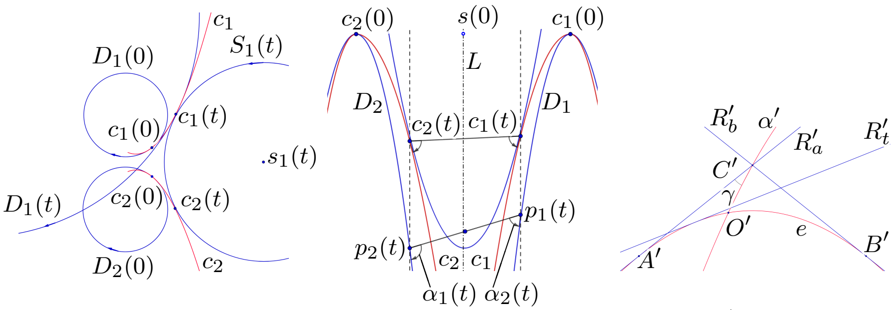

Fig. 3: The notation in the proofs of Lemmas 7, 13, and 19 (from the left to the right).

Next, let us prove that $S_1(t)$ and $D_1(0)$ are disjoint. For this purpose, we compute the derivative of the signed curvature $k_1(t)$ of the curve $c_1(t)$ at $t=0$. By the Euler and Meusnier theorems, we get

$$k_1(t) = \frac{\kappa_1(t)\cos^2\angle(c_1, c(t)) + \kappa_2(t)\sin^2\angle(c_1, c(t))}{\cos\angle(s(t)c_1(t), P)}.$$

Differentiating, we get $k_1'(0) = \kappa_2'(0) \neq 0$ because $\angle(c_1, c(0)) = \pi/2$ and $\angle(s(0)c_1(0), P) = 0$. Since the signed curvatures of $S_1(0)$ and $D_1(0)$ are $\kappa_1(0)$ and $\kappa_2(0)$ respectively, by the assumption $\kappa_2'(0)(\kappa_1(0) - \kappa_2(0)) > 0$ it follows that the signed curvature of $D_1(t)$ is between the signed curvatures of $D_1(0)$ and $S_1(t)$ for $t > 0$ small enough. By the Tait–Kneser theorem, $D_1(t)$ is disjoint from $D_1(0)$, and by construction, $D_1(t)$ has an oriented contact with $S_1(t)$. Thus $D_1(0)$ and $S_1(t)$ are separated by $D_1(t)$, hence are disjoint.

Finally, we estimate the distance from $s_1(t)$ to the bisector of $c_1(0)c_2(0)$. Since c_1 and $D_1(0)$ have second-order contact, it follows that $c_1(t)$ is within distance $O(t^3)$ from $D_1(0)$. Then the distance between the disjoint circles $S_1(t)$ and $D_1(0)$ is $O(t^3)$. Analogously, the distance between $S_1(t)$ and $D_2(0)$ is $O(t^3)$. Then the difference in the distances from the center $s_1(t)$ to the centers of $D_1(0)$ and $D_2(0)$ is $O(t^3)$. Since $D_1(0)$ and $D_2(0)$ are symmetric with respect to the bisector of $c_1(0)c_2(0)$, it follows that the distance from $s_1(t)$ to the bisector is $O(t^3)$, which proves the lemma. □

4.2 Surfaces with both isotropic principal curvatures constant

As a motivation for studying isotropic channel surfaces, and also as one step in their classification, let us find all surfaces with both isotropic principal curvatures constant.

Theorem 8. *An admissible surface has constant nonzero isotropic principal curvatures κ_1 and κ_2 if and only if it is isotropic congruent to a subset of the paraboloid $2z = \kappa_1 x^2 + \kappa_2 y^2$.*

Let us see how the notion of an isotropic channel surface naturally arises in the proof of this theorem.

Lemma 9. *Assume that the isotropic principal curvature κ_1 along an isotropic principal curvature line c of an admissible surface C is constant and nonzero. At each point of c, take the parabolic isotropic sphere with the radius $1/\kappa_1$ that is tangent to C at this point. Then all these isotropic spheres coincide.*

Proof By the *center* of a parabolic isotropic sphere of radius r we mean the point obtained from the vertex of the paraboloid by the translation by the vector $(0,0,r)$. The unique parabolic isotropic sphere of radius r with the center $m = (m_1, m_2, m_3)$ is given by the equation

$$2(z - m_3 + r) = \frac{1}{r}\left((x-m_1)^2 + (y-m_2)^2\right). \tag{5}$$

Beware that the notion of a center is not invariant under isotropic congruences, but similar notions are common in isotropic geometry [8, Chapter 9].

The *isotropic curvature center* of C at a point $p = (x, y, z) \in c$ is the center of the parabolic isotropic sphere of radius $r = 1/\kappa_1$ that is tangent to C at p. It is given by

$$m = \begin{bmatrix} x - \frac{1}{\kappa_1} f_x \\ y - \frac{1}{\kappa_1} f_y \\ z - \frac{1}{2\kappa_1}\left(f_x^2 + f_y^2 - 2\right) \end{bmatrix},$$

where we locally represent C as the graph of a function $z = f(x, y)$. This formula is obtained from the three equations (5), $f_x = \kappa_1(x - m_1)$, $f_y = \kappa_1(y - m_2)$.

Now let $p(t) = (x(t), y(t), f(x(t), y(t)))$ run through the isotropic principal curvature line c and let $m(t)$ be the corresponding isotropic curvature center. It suffices to prove that $m'(t) = 0$. We omit the arguments of the functions x, y, f in what follows.

Since c is the isotropic principal curvature line, it follows that

$$\begin{cases} f_{xx}x' + f_{xy}y' = \kappa_1 x' \\ f_{xy}x' + f_{yy}y' = \kappa_1 y', \end{cases}$$

hence

$$m' = \begin{bmatrix} x' - \frac{1}{\kappa_1} f_{xx}x' - \frac{1}{\kappa_1} f_{xy}y' \\ y' - \frac{1}{\kappa_1} f_{xy}x' - \frac{1}{\kappa_1} f_{yy}y' \\ f_x x' + f_y y' - f_x x' - f_y y' \end{bmatrix} = 0.$$

□

The meaning of this lemma is that $\kappa_1 = \text{const}$ implies that the surface is essentially an isotropic pipe surface, which we are going to define now.

An *isotropic channel surface* is the envelope of a smooth one-parameter family of parabolic isotropic spheres $S(t)$, i.e. the surface C tangent to each isotropic sphere $S(t)$ along a single curve $c(t)$ without endpoints so that the curves $c(t)$ cover C. The curves $c(t)$ are called *characteristics*. If the radii of the isotropic spheres are constant, then C is called an *isotropic pipe surface*.

To proceed, we need an isotropic analog of Lemma 6 above.

Lemma 10. *Consider a characteristic $c(t)$ of an isotropic channel surface C. Let $r(t)$ be the radius of the isotropic sphere $S(t)$ that is tangent to C along $c(t)$. Then $c(t)$ is an isotropic principal curvature line on C and the isotropic principal curvature along $c(t)$ is $1/r(t)$. If $r'(t) \neq 0$ then $c(t)$ is an elliptic isotropic circle. If $r(t) = \text{const}$ then for some t the curve $c(t)$ is a parabolic isotropic circle.*

Proof Since $c(t)$ is an isotropic principal curvature line of $S(t)$, then by the isotropic Joachimsthal theorem [6, Section 36], it is an isotropic principal curvature line of C.

Let p be a point on $c(t)$ and T be the tangent line to C through p. Then by Meusnier's theorem (Theorem 2), the osculating isotropic circle of $c(t)$ lies on the isotropic sphere of radius $1/\kappa_n$ that is tangent to C at p. Here κ_n is the isotropic principal curvature along $c(t)$ because $c(t)$ is an isotropic principal curvature line.

On the other hand, $c(t)$, hence its osculating isotropic circle, lies on the isotropic sphere $S(t)$ of radius $r(t)$ that is also tangent to C at p. This implies that $\kappa_n = 1/r(t)$.

Now let $S(t)$ have the equation

$$2z = A(t)\left(x^2 + y^2\right) + B(t)x + C(t)y + D(t).$$

Then $c(t)$ is contained in the set defined by the system (cf. [50, Section 5.13])

$$\begin{cases} A(t)\left(x^2+y^2\right)+B(t)x+C(t)y+D(t)-2z=0, \\ A'(t)\left(x^2+y^2\right)+B'(t)x+C'(t)y+D'(t)=0. \end{cases} \tag{6}$$

If $r'(t) \neq 0$ then $A'(t) = r'(t)/r(t)^2 \neq 0$ because $r(t) = 1/A(t)$. Remove the quadratic terms in the second equation by subtracting the first equation with the coefficient $A'(t)/A(t) \neq 0$. This introduces a term linear in z into the second equation. Thus $c(t)$ is the intersection of $S(t)$ with a non-isotropic plane, hence an elliptic isotropic circle.

Finally, assume that $r(t) = \text{const}$ so that $A'(t) = 0$. There exists t such that at least one of the derivatives $B'(t), C'(t), D'(t) \neq 0$, otherwise all $S(t)$ coincide and there is no envelope. Then the second equation of (6) defines an isotropic plane. Thus $c(t)$ is the intersection of $S(t)$ with the plane, hence a parabolic isotropic circle. □

This lemma and its proof remain true, if we allow the characteristics $c(t)$ to have endpoints (in the definition of an isotropic channel surface); then $c(t)$ is going to be an arc of an isotropic circle instead of a full one.

Proof of Theorem 8 By Lemma 9, locally there are two families of congruent parabolic isotropic spheres of radii $1/\kappa_1$ and $1/\kappa_2$ respectively that are tangent to the surface along isotropic principal curvature lines. By the last assertion of Lemma 10, one of the characteristics $c(t)$ of one family is an arc of a parabolic isotropic circle. Performing an appropriate isotropic congruence, one can make the parabolic isotropic sphere $S(t)$ that is tangent to the surface along $c(t)$ symmetric with respect to the plane of $c(t)$. The isotropic spheres of the other family are congruent and touch $S(t)$ at the points of $c(t)$. Hence they are obtained by parabolic rotations of one isotropic sphere, and thus the surface is locally parabolic rotational. By Proposition 4, the theorem follows. (The envelope of the other family can also be computed directly.) □

4.3 Isotropic channel CRPC surfaces

The classification of isotropic channel CRPC surfaces is similar to the Euclidean ones, but in addition to rotational surfaces, we get parabolic rotational ones.

An *analytic isotropic channel surface* is an analytic surface that is the envelope of a one-parameter analytic family of parabolic isotropic spheres.

Theorem 11. *An analytic isotropic channel Weingarten surface is an isotropic rotational or isotropic pipe surface. In particular, an analytic isotropic channel surface with a constant nonzero ratio of isotropic principal curvatures is a subset of an isotropic rotational or parabolic rotational one.*

The proof is analogous to the Euclidean one, but we consider the centers curve instead of the spine curve. If the characteristics $c(t)$ are elliptic isotropic circles, then the locus of their centers $s(t)$ is called the *centers curve*.

Lemma 12. *If the centers curve of an analytic isotropic channel surface is contained in a vertical line, then the surface is isotropic rotational.*

Proof Bring the vertical line to the z-axis by an appropriate isotropic congruence. Use the notation from the proof of Lemma 10. If $A'(t) \neq 0$, then the second equation of (6) defines a circle, which represents the top view of the characteristic $c(t)$. Thus the top view of all the characteristics $c(t)$ are circles with the center at the origin. Therefore $B'(t) = C'(t) = 0$ for all t, hence $B(t)$ and $C(t)$ are constants. Then the second equation of (6) takes form $x^2 + y^2 = -D'(t)/A'(t)$. Substituting it into the first equation of (6), we obtain that all the isotropic circles $c(t)$ lie in the planes parallel to one plane $B(0)x + C(0)y - 2z = 0$. By continuity, this remains true for the roots of the equation $A'(t) = 0$. Therefore the surface is isotropic rotational. □

To ensure that the centers curve $s(t)$ is a vertical line, we need the following two technical lemmas. In the first one, for a line segment joining two symmetric parabolic isotropic circles, we express the distance from the midpoint of the segment to the symmetry axis in terms of the replacing angles between the segment and the isotropic circles. The *(oriented) replacing angle* between two non-isotropic lines lying in one isotropic plane is the difference in their slopes.

Lemma 13. *(See Fig. 3 to the middle.) Assume that two parabolic isotropic circles D_1 and D_2 of isotropic curvature A lie in an isotropic plane and are symmetric with respect to an isotropic line L. Let the distance between their axes be $2B \neq 0$. Let two points p_1 and p_2 lie on D_1 and D_2 respectively. Let the segment p_1p_2 have isotropic length d and form replacing angles α_1 and α_2 with D_1 and D_2 respectively. Then the isotropic distance from the midpoint of p_1p_2 to the line L equals $|\alpha_1 + \alpha_2|d/|8AB|$.*

Proof Without loss of generality, D_1 and D_2 lie in the xz-plane and have the equations $z = A(x \pm B)^2$. If the x-coordinates of p_1 and p_2 are x_1 and x_2, then we compute directly $|\alpha_1 + \alpha_2| = 4|AB| \cdot |x_1 + x_2|/|x_1 - x_2|$. Since $|x_1 - x_2| = d$ and $|x_1 + x_2|/2$ is the desired isotropic distance, the lemma follows. □

Lemma 14. *Under the notation of Lemma 10, assume that the second principal curvature is constant along $c(t)$ and different from the first one. If $r'(t) \neq 0$, then $s'(t)$ is vertical (or zero).*

Proof Fix a particular value of t, say, $t = 0$, and assume that t is sufficiently close to this value. It suffices to prove that the isotropic distance from $s(t)$ to $s(0)$ is $O(t^2)$.

We do it by reduction to a planar problem. See Fig. 3 to the middle. Since $r'(0) \neq 0$, by Lemma 10 it follows that $c(0)$ is an elliptic isotropic circle. Performing an isotropic congruence of the form $z \mapsto z + px + qy$, we can take $c(0)$ to a horizontal circle. Take an isotropic plane P passing through $s(0)$ and parallel to $s'(0)$. The section of $c(t)$ by the plane P consists of two points $c_1(t)$ and $c_2(t)$. The section of C coincides with the two curves $c_1(t)$ and $c_2(t)$ because the characteristics cover the surface. The section of $S(t)$ is a parabolic isotropic circle that is tangent to the two curves at the points $c_1(t)$ and $c_2(t)$. Then the chord $c_1(t)c_2(t)$ forms equal replacing angles (of opposite signs) with the tangents at $c_1(t)$ and $c_2(t)$. It suffices to prove that the isotropic distance from the midpoint of the chord to $s(0)$ is $O(t^2)$.

Let us estimate how those replacing angles and the midpoint change if we replace the two curves with their osculating isotropic circles D_1 and D_2 (possibly degenerating into lines) at the points $c_1(0)$ and $c_2(0)$. Since the second principal curvature is constant along $c(0)$ and the plane of $c(0)$ is horizontal, it follows that D_1 and D_2 are symmetric with respect to the vertical line L through $s(0)$. Since the first principal curvature is different, it follows that $D_1 \neq D_2$. Let $p_1(t)$ and $p_2(t)$ be the points of D_1 and D_2 lying on the vertical lines through the points $c_1(t)$ and $c_2(t)$ respectively. The replacing angle between the tangents to the respective curves at the points $p_1(t)$ and $c_1(t)$ is $O(t^2)$ because D_1 is osculating. The same is true for the tangents at $p_2(t)$ and $c_2(t)$. The replacing angle between $p_1(t)p_2(t)$ and $c_1(t)c_2(t)$ is $O(t^3)$. Thus the replacing angles $\alpha_1(t)$ and $\alpha_2(t)$ which $p_1(t)p_2(t)$ forms with D_1 and D_2 satisfy $\alpha_1(t) + \alpha_2(t) = O(t^2)$. Now the result follows from Lemma 13 and its analog for lines D_1 and D_2. □

Proof of Theorem 11 Consider an analytic isotropic channel Weingarten surface C and the isotropic parabolic sphere $S(t)$ of radius $r(t)$ that is tangent to C along the characteristic $c(t)$. Then according to Lemma 10, the isotropic principal curvature κ_1 along $c(t)$ is $1/r(t)$. If the Weingarten relation is not $\kappa_1 = \text{const}$, then the other isotropic principal curvature κ_2 is constant along $c(t)$.

If $r(t) = \text{const}$ or $\kappa_1 \equiv \kappa_2$, then we get an isotropic pipe surface or an isotropic sphere. Otherwise, restrict the range of t to an interval where $r'(t) \neq 0$ and $\kappa_1 \neq \kappa_2$. By Lemma 14, the centers curve $s(t)$ has vertical tangent vector $s'(t)$ for all t. Then $s(t)$ is contained in a vertical line, and by Lemma 12 the surface C is isotropic rotational.

Finally, suppose that C has a constant ratio of the isotropic principal curvatures. Then it is Weingarten. As we have proved, it is an isotropic rotational or pipe surface. In the latter case, $\kappa_1 = \text{const}$ by Lemma 10, hence $\kappa_2 = \text{const}$, and C is a subset of a parabolic rotational surface by Theorem 8. □

5 Ruled surfaces

Let us now turn to ruled surfaces. In Euclidean geometry we will not encounter a new surface, but in isotropic geometry there is a non-trivial CRPC ruled surface.

Our arguments are based on line geometry. For the concepts used in the following, we refer to [52]. The methods for ruled surfaces and channel surfaces are actually related via Lie's line-sphere correspondence. We again restrict ourselves to analytic surfaces (with nonvanishing Gaussian curvature); then the rulings form an analytic family because the direction of a ruling is asymptotic.

An *analytic ruled surface* is an analytic surface covered by an analytic family of line segments. The lines containing the segments are the *rulings*.

5.1 Euclidean ruled CRPC surfaces.

We start with the Euclidean case and show the following result.

Proposition 15. *The only ruled surfaces with a constant nonzero ratio of principal curvatures are the ruled minimal surfaces, i.e., helicoids.*

Proof A ruled CRPC surface must be skew (without torsal rulings) and its asymptotic curves should intersect under a constant angle γ [20, Section 2.1]. One family of asymptotic curves is the rulings. Let us fix a ruling R and consider the other asymptotic tangents $A(p)$ (different from R) at all points $p \in R$. They form a quadric $L(R)$ (the so-called *Lie quadric* of R; see e.g. [52, Corollary 5.1.10]). If the angle between R and $A(p)$ is constant, it can only be a right angle (so that $L(R)$ is a right hyperbolic paraboloid). Indeed, if the angle γ is not a right one, then the ideal points of the lines $A(p)$ form a conic c_ω in the ideal plane ω (ideal conic of a rotational cone with axis R) which does not contain the ideal point R_ω of R. However, R_ω and c_ω should lie in the same curve $L(R) \cap \omega$ (conic or pair of lines), which is not possible. So, $\gamma = \pi/2$ and our surface is a skew ruled minimal surface, i.e. a helicoid by the Catalan theorem. □

Remark 16. Here we used the famous Catalan theorem stating that the only ruled minimal surfaces are helicoids and planes. Remarkably, in Lemmas 18–20 below we actually obtain a line-geometric proof of this classical result. Indeed, we have just shown that the Lie quadric of each ruling must be a right hyperbolic paraboloid. Then Lemma 18 and its proof remain true in Euclidean geometry. Then without loss of generality, all the rulings are parallel to the plane $z = 0$. Since the asymptotic directions and the rulings are orthogonal, their top views are also orthogonal, and our Euclidean minimal surface is an *isotropic* minimal surface as well. The Catalan theorem now reduces to Lemmas 19–20, where the case of a hyperbolic paraboloid is easily excluded.

The proof of Proposition 15 already indicates that there is hope to get a ruled CRPC surface to a constant $a \neq -1$ in isotropic geometry. This is what we will now pursue.

5.2 Isotropic ruled CRPC surfaces.

Theorem 17. *(See Fig. 4) An admissible analytic ruled surface has a constant ratio $a < 0$ of isotropic principal curvatures if and only if it is isotropic similar to a subset of either the hyperbolic paraboloid $z = x^2 + ay^2$, or the helicoid*

$$r(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ v \end{bmatrix}, \qquad \text{if } a = -1, \tag{7}$$

or the surface

$$r_a(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ \exp\left(\frac{a+1}{\sqrt{|a|}}v\right) \end{bmatrix}, \qquad \text{if } a \neq -1. \tag{8}$$

Proof This follows directly from Lemmas 18–20 below (which themselves rely on standard Lemmas 31–32 from Appendix A). □

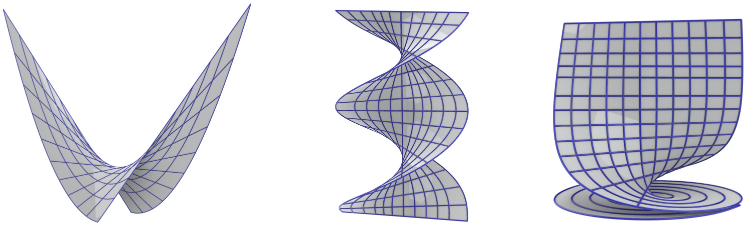

Fig. 4: Ruled isotropic CRPC surfaces (from the left to the right): a hyperbolic paraboloid, helicoid (7), and spiral ruled surface (8).

Lemma 18. *An admissible analytic ruled surface with a constant nonzero ratio of isotropic principal curvatures is a* conoidal *(or* Catalan*) surface, i.e. all the rulings are parallel to one plane.*

Proof Let us show that the Lie quadric of each ruling R (see [52, Corollary 5.1.10]) is a hyperbolic paraboloid. Since the surface is admissible, the top view R' of R is not a point. Since isotropic angles are seen as Euclidean angles in the top view, the top views $A(p)'$ of the asymptotic tangents $A(p)$ at points p of R must form the same angle γ with R'. Hence the lines $A'(p)$ are parallel to each other, implying that the quadric formed by the lines $A(p)$ is a hyperbolic paraboloid.

Then the second family of rulings (distinct from $A(p)$) of the quadric intersects the ideal plane ω by a line A_ω. Since the Lie quadric has a second-order contact with our surface [52, Theorem 5.1.9], it follows that A_ω has a second-order contact with the ideal curve s_ω formed by the ideal points of the rulings of our surface [52, Proposition 5.1.11]. As a curve that has an osculating straight line at each point, the curve s_ω is itself a straight line and therefore our surface must be a conoidal surface (s_ω cannot degenerate to a point as the isotropic Gaussian curvature $K \neq 0$). □

Lemma 19. *An admissible analytic conoidal surface with a constant nonzero ratio of isotropic principal curvatures is a* conoid*, i.e., all the rulings are parallel to a fixed plane and intersect a fixed line. The fixed line is either vertical or belongs to another family of rulings. In the latter case, the surface is a hyperbolic paraboloid with a vertical axis.*

Proof By the analyticity, it suffices to prove the lemma for an arbitrarily small part of our surface. Thus in what follows we freely restrict and extend our surface.

Let R_t be the analytic family of the rulings of the surface. Since $K \neq 0$, it follows that there are no torsal rulings; in particular, the surface is not a plane.

Let R'_t be the top view of R_t. By Lemma 31, one of the following cases (i)–(iii) holds, after we restrict t to a smaller interval.

Case (i): all R'_t have a common point. Then all R_t intersect one vertical line and the lemma is proved.

Case (ii): all R'_t are parallel. Then due to the fixed angle between the asymptotic directions in the top view, the second family of asymptotic curves also appears as parallel lines in the top view. Hence those curves lie in the isotropic planes. However, at non-inflection points of the asymptotic curves the osculating planes are the tangent planes of the surface [53, Page 28]. Thus these tangent planes needed to be isotropic, which is not possible for an admissible surface. Hence, there are no non-inflection points, both families of asymptotic curves are straight lines, and our surface is a hyperbolic paraboloid with a vertical axis.

Case (iii): all R'_t touch one curve e (envelope). Let us show that this case is actually impossible.

For this purpose, we are going to extend our surface to reach the envelope. Let $h(t)$ be the Euclidean distance from the ruling R_t to the fixed plane parallel to all the rulings. We have $h(t) \neq \text{const}$ because our surface is not a plane. By continuity, there is an interval I where $h'(t)$ has a constant sign. Then the union $\bigcup_{t\in I} R_t$ is an analytic surface containing a part of the initial surface. The resulting surface is not admissible: By Lemma 31 the envelope forms a part of the boundary of the top view of the surface, hence the tangent planes are isotropic at the points with the top views lying on the envelope.

Switch to the new surface $\bigcup_{t\in I} R_t$. By analyticity, it still has a constant ratio of isotropic principal curvatures (at the admissible points). The Gaussian curvature still vanishes nowhere because there are no torsal rulings. Thus the whole surface, including non-admissible points, is covered by two analytic families of asymptotic curves (recall that the asymptotic curves are the same in Euclidean and isotropic geometry, hence they acquire no singularities at the non-admissible points). One of the families consists of the rulings, and at admissible points, the other one crosses them under constant angle γ in the top view.

Now we prove that there is an asymptotic curve α containing a non-admissible point O but not entirely consisting of non-admissible points. See Fig. 3 to the right. The top view α' needs to have a common point with the envelope e. Take $a, b \in I$ close enough so that the angle between R'_a and R'_t is an increasing function in t on $[a, b]$ not exceeding $\pi - \gamma$. Let A' and B' be the tangency points of R'_a and R'_b with the envelope, and $C \in R_a$ be the point with the top view $C' := R'_a \cap R'_b$. Then the second asymptotic curve α through C is the required one. Indeed, the angle between its top view α' and R'_a equals γ, hence α' enters the curvelinear triangle $A'B'C'$ formed by the arc AB of the envelope and two straight line segments $B'C'$ and $C'A'$. Since the angle between α' and R'_t is constant and the angle between R'_a and R'_t is increasing, the curve α' cannot reach the sides $B'C'$ and $C'A'$ as long as it remains smooth. Since the asymptotic curves extend till the surface boundary, it follows that α' has a common point with the envelope and α has a non-admissible point O, as required.

Since α does not entirely consist of non-admissible points, it follows that at the other close enough points $P \neq O$ of α, the tangents cross the rulings under constant angle γ in the top view. By the continuity, the limit L' of the top views of the tangents crosses the top view of the ruling through O under angle γ.

Now let us prove that L' must be the top view of the ruling through O, and thus get a contradiction.

If the tangent of α at O is not vertical, then L' coincides with the top view of the tangent, hence with the top view of the tangent plane at O, hence with the top view of the ruling through O.

If the tangent of α at O is vertical, then by Lemma 32 the limit L' coincides with the top view of the limit of the osculating planes at the points of α. But the osculating plane of an asymptotic curve at non-inflection points is the tangent plane, and the points close enough to O are non-inflection. Hence we again obtain the top view of the tangent plane at O, equal to the top view of the ruling through O.

This contradiction shows that case (iii) is impossible, completing the proof. $\square$

Lemma 20. *An admissible conoid has a constant ratio $a \neq 0$ of isotropic principal curvatures if and only if it is isotropic similar to a subset of one of the surfaces $z = x^2 + ay^2$, (7), or (8) from Theorem 17.*

Proof Let all the rulings of the conoid be parallel to a fixed plane α and intersect a fixed line l. We have two possibilities indicated in Lemma 19.

If $l \nparallel Oz$ then by Lemma 19 the surface is a hyperbolic paraboloid with a vertical axis. By Example 1 it is isotropic similar to a subset of the paraboloid $z = x^2 + ay^2$.

If $l \parallel Oz$ then performing an isotropic similarity, we can take l to the z-axis and α to the plane $z = 0$. The resulting conoid can be parameterized as

$$r(u, v) = (u \cos v, u \sin v, h(v))$$

for some smooth function $h(v)$. The asymptotic curves (distinct from the rulings) are characterized by the differential equation [54, p. 137]

$$u(v)^2 = bh'(v)$$

for some constant b. The top views of the asymptotic curves must intersect the lines through the origin under the constant angle γ, where $\cot^2(\gamma/2) = |a|$. We now have to distinguish whether this angle is right one or not.

If γ is a right angle, then the top views of asymptotic curves must be concentric circles, leading to $u(v) = \text{const}$ and $h(v) = v$ up to a translation and a scaling along the z-axis. We get helicoid (7).

If γ is not a right angle, then the top views must be logarithmic spirals $u(v) = ce^{\pm v \cot \gamma}$ for some constant c. This yields $h(v) = e^{\pm 2v \cot \gamma}$ up to a translation and a scaling along the z-axis. Changing the signs of v and y, if necessary, we arrive at (8). □

5.3 Geometry of the surfaces and their characteristic curves

Ruled CRPC surface (8) is a *spiral surface* (see [55]), generated by a one-parameter group of (Euclidean and isotropic) similarities, composed of rotations about the z-axis and central similarities with center at the origin. The paths of that motion are cylindro-conical spirals which appear in the top view as logarithmic spirals with polar equation $r(v) = c \cdot e^{2v \cot \gamma}$ for some constant c. However, the asymptotic curves (different from rulings) are not such paths. They are expressed as

$$c(v) = (ce^{v \cot \gamma} \cos v, ce^{v \cot \gamma} \sin v, e^{2v \cot \gamma}),$$

and are also obtained by intersecting the ruled surface with isotropic spheres (of variable isotropic radius $c^2/2$),

$$z = \frac{1}{c^2}(x^2 + y^2). \tag{9}$$

On these, the curves $c(v)$ are isotropic loxodromes. Their tangents are contained in a linear line complex with the z-axis as the axis. This is related to another non-Euclidean interpretation of isotropic CRPC ruled surface (8) and its asymptotic curves $c(v)$: One can view one of the paraboloids (9) as absolute quadric of the projective model of hyperbolic 3-space. There, (8) is a helicoid and the asymptotic curves $c(v)$ are paths of a hyperbolic helical motion (one-parameter subgroup of the group of hyperbolic congruence transformations). It is also well known and easy to see that the hyperbolic helices are projectively equivalent to Euclidean spherical loxodromes [56].

In summary, we have proved the following result.

Proposition 21. *The asymptotic curves of spiral ruled surfaces* (8)*, distinct from the rulings, lie on isotropic spheres. Viewing one of these isotropic spheres as the absolute quadric of the projective model of hyperbolic geometry, these surfaces are helicoids and the asymptotic curves are helical paths. The latter are projectively equivalent to Euclidean spherical loxodromes.*

6 Helical surfaces

A *helical motion through the angle ϕ about the z-axis with pitch h* is the composition of the rotation through the angle ϕ about the z-axis and the translation by $h\phi$ in the z-direction. The helical motion is also an isotropic congruence. A surface invariant under the helical motions for fixed h and all ϕ is called *helical with pitch h*. In particular, for $h = 0$ we get a rotational surface.

Theorem 22. *(See Fig. 5) An admissible helical surface with nonzero pitch has a constant ratio $a \neq 0$ of isotropic principal curvatures, if and only if it is isotropic similar to a subset of one of the surfaces*

$$r_a(u,v) = \begin{bmatrix} \cos v\,(\cos u \sin^a u)^{-\frac{1}{a+1}} \\ \sin v\,(\cos u \sin^a u)^{-\frac{1}{a+1}} \\ v + u + \frac{1}{a^2-1}\tan u + \frac{a^2}{a^2-1}\cot u \end{bmatrix}, \qquad \text{if } a \neq \pm 1, \tag{10}$$

$$r_c(u,v) = \begin{bmatrix} u\cos v \\ u \sin v \\ c\log u + v \end{bmatrix}, \qquad \text{if } a = -1, \tag{11}$$

where c is an arbitrary constant and v runs through $\mathbb{R}$. In (11)*, u runs through* $(0, +\infty)$*. In* (10)*, u runs through a subinterval of* $(0, \pi/2)$*, where* $\tan^2 u \neq a$*.*

Proof Since the pitch is nonzero, it can be set to 1 by appropriate scaling along the z-axis. Take the section of the surface by a half-plane bounded by the z-axis. Since the surface is admissible, the section is a disjoint union of smooth curves without vertical tangents. Then one of those curves can be parameterized as $z = f(\sqrt{x^2+y^2})$ for some smooth function $f(u)$ defined in an interval inside the ray $u > 0$. Hence, up to rotation about the z-axis, our surface can be parameterized as

$$r(u,v) = (u\cos v, u\sin v, f(u) + v). \tag{12}$$

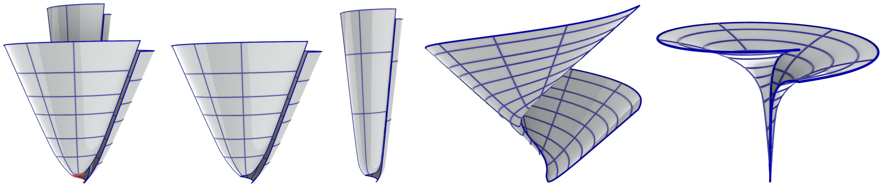

Fig. 5: Helical isotropic CRPC surfaces (from the left to the right): surface (10) for $a>0$; its outer part; its inner part; surface (10) for $a<0$; surface (11). The singular curve $\tan^2(u)=a$ of the leftmost surface is depicted in red; it splits the surface into the parts $\tan^2(u)>a$ and $\tan^2(u)<a$ shown separately.

Then the isotropic Gaussian and mean curvatures are (see [57, Eq. (4.4)])

$$K=\frac{u^3f''(u)f'(u)-1}{u^4},\qquad H=\frac{f'(u)+uf''(u)}{2u}. \tag{13}$$

Then the equation $H^2/K=(a+1)^2/(4a)$ is equivalent to

$$au^2(f'(u)+uf''(u))^2=(a+1)^2(u^3f''(u)f'(u)-1). \tag{14}$$

First let us solve the equation for $a=-1$. In this case, $f''(u)u+f'(u)=0$. Hence $f(u)=c\log u+c_1$ for some constants c and c_1. By performing the isotropic similarity $z\mapsto z-c_1$ we bring our surface to form (11).

Assume further that $a\neq -1$. Then (14) is equivalent to (see [49, Section 1.1])

$$\left(\frac{(a-1)\left(u^2f''(u)+uf'(u)\right)}{2(a+1)}\right)^2-\left(\frac{uf'(u)-u^2f''(u)}{2}\right)^2=1. \tag{15}$$

Thus the first fraction here vanishes nowhere (in particular, $a\neq 1$). We may assume that it is positive, otherwise change the sign of f and v in (12), leading to just a rotation of the surface through the angle π about the x-axis. Then the first fraction in (15) can be set to $\csc(2s(u))$ and the second one can be set to $\cot(2s(u))$ for some smooth function $s(u)$ with the values in $(0,\pi/2)$. Therefore by direct calculations (see [49, Section 1.2])

$$f'(u)=\frac{a\cot s(u)+\tan s(u)}{(a-1)u}, \tag{16}$$

$$f''(u)=\frac{a\tan s(u)+\cot s(u)}{(a-1)u^2}. \tag{17}$$

Taking the derivative of (16) with respect to u and combining it with (17) we obtain

$$\frac{s'(u)(\tan s(u)-a\cot s(u))}{(a+1)}=\frac{1}{u} \tag{18}$$

(see [49, Section 1.3]). In particular, $s'(u)\neq 0$ everywhere, hence $s(u)$ has an inverse function $u(s)$. Integrating both sides of (18) and using that $s(u)$ assumes values in $(0,\pi/2)$, we get $u(s)=c_2\,(\cos s\sin^a s)^{-\frac{1}{a+1}}$ for some constant $c_2\neq 0$.

Denote $f(s):=f(u(s))$. By the chain rule, (16), and (18) we get

$$f'(s)=\left.\frac{f'(u)}{s'(u)}\right|_{u=u(s)}=\frac{(\tan s+a\cot s)(\tan s-a\cot s)}{(a-1)(a+1)}. \tag{19}$$

Integrating both sides of (19), we get

$$f(s) = s + \cot 2s + \frac{a^2+1}{a^2-1}\csc 2s + c_3 = s + \frac{1}{a^2-1}\tan s + \frac{a^2}{a^2-1}\cot s + c_3$$

for some constant c_3 [49, Section 1.4]. The isotropic similarity $(x, y, z) \mapsto (x/c_2, y/c_2, z - c_3)$ brings our surface to form (10) (up to renaming the parameter s to u). $\square$

Family (11) is a family of helical isotropic minimal surfaces joining helicoid (7) and logarithmoid (3) (after appropriate scaling of the z-coordinate). It can be alternatively described as the family of the graphs of the harmonic functions $z = \mathrm{Re}(C \log(x + iy))$ with varying complex parameter C (again, up to isotropic similarity). It is the associated family of the helicoid in isotropic geometry [47, Sections 4.1 and 4.3(a)]; thus surfaces (11) can be called "*isotropic helicatenoids*". Just like their Euclidean analogs, they are isometric to each other for different c, after scaling of the z-coordinate by $1/\sqrt{1+c^2}$. Notice that in isotropic geometry, the most natural notion of *isometry* requires preservation of both the metric and the isotropic Gaussian curvature [58]. This is indeed the case here by (13).

7 Translational surfaces

Now we present the main result of the paper. If $\alpha(u)$ and $\beta(v)$ are two curves in $\mathbb{R}^3$, then the surface $r(u, v) = \alpha(u) + \beta(v)$ is called the *translational surface formed by $\alpha(u)$ and $\beta(v)$*.

Theorem 23. *(Fig. 6) An admissible translational surface formed by a planar curve α and another curve β has a constant ratio $a \neq 0$ of isotropic principal curvatures, if and only if it is isotropic similar to a subset of one of the surfaces*

$$r_a(u, v) = \begin{bmatrix} u \\ v \\ v^2 + au^2 \end{bmatrix}, \tag{20}$$

$$r_b(u, v) = \begin{bmatrix} v + b\cos v \\ b\sin v + (b^2-1)\log|b - \sin v| + (1-b^2)u \\ \exp u \end{bmatrix}, \quad \text{if } a \neq 1, \tag{21}$$

$$r(u, v) = \begin{bmatrix} u + v \\ \log|\cos u| - \log|\cos v| \\ u \end{bmatrix}, \quad \text{if } a = -1, \tag{22}$$

where we denote $b := (a+1)/(a-1)$. In particular, β must be a planar curve as well. In (20), the variables u, v run through $\mathbb{R}$. In (21), u runs through $\mathbb{R}$ and v runs through an interval where $\sin v \neq b$ (for $a < 0$) and $b \sin v \neq 1$ (for $a > 0$). In (22), (u, v) runs through a subdomain of $(-\pi/2, \pi/2)^2 \setminus \{u + v = 0\}$.

Surfaces (20)–(22) for $a = -1$ can be viewed as the three *isotropic Scherk minimal surfaces* [42, 47].

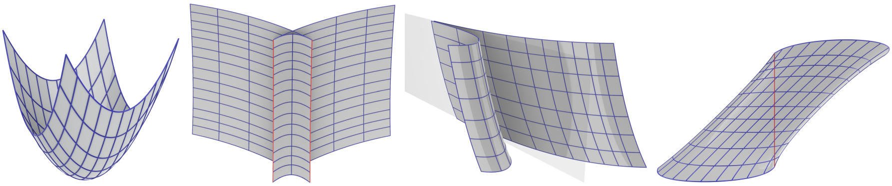

Fig. 6: Translational isotropic CRPC surfaces (from the left to the right): surface (20) for $a > 0$; surface (21) for $a > 0$ and $a < 0$; surface (22). The red curves of surface (21) are the singular curves $b \sin(v) = 1$. The red line of (22) is the line $u + v = 0$ where the surface has isotropic tangent planes. The transparent plane is the asymptotic plane of surface (21) with $a < 0$ obtained in the limit $v \to \arcsin b$.

The theorem follows from Lemmas 24–28, where the following 5 cases are considered:

1. α and β are isotropic planar;
2. α is isotropic planar and β is non-isotropic planar;
3. α and β are non-isotropic planar;
4. α is isotropic planar and β is non-planar;
5. α is non-isotropic planar and β is non-planar.

In our arguments, we use the expressions for K and H obtained by combining [7, Eq. (4,8), (4,11), (4,18), (5,1)]. (The convention for the sign of H in [7] is different from ours but this does not affect the equation $H^2/K = \frac{(a+1)^2}{4a}$ of CRPC surfaces.)

Lemma 24. *Under the assumptions of Theorem 23, if α and β are contained in isotropic planes, then the surface is isotropic similar to a subset of* (20).

Proof Performing a rotation about a vertical axis, we can take the planes of α and β to the planes $y = kx$ and $x = 0$ for some $k \in \mathbb{R}$. Since the surface is admissible, α and β cannot have vertical tangents. Thus the curves can be parameterized as $\alpha(u) = (u, ku, f(u))$ and $\beta(v) = (0, v, g(v))$ for some functions $f(u)$ and $g(v)$ defined on some intervals. Then our surface is

$$r(u, v) = (u, ku + v, f(u) + g(v)). \tag{23}$$

Then the isotropic Gaussian and mean curvatures are (see [7, Sections 4–5] and [59, Eq. (3.2)])

$$K = f''(u)g''(v), \quad H = \frac{(k^2 + 1)\, g''(v) + f''(u)}{2}. \tag{24}$$

Since $K \neq 0$, it follows that $f''(u)g''(v) \neq 0$ for all (u, v) from the domain. Then the equation $H^2/K = (a + 1)^2/(4a)$ is equivalent to the following differential equation:

$$\left(k^2 + 1 + \frac{f''(u)}{g''(v)}\right)^2 = \frac{(a + 1)^2}{a}\ \frac{f''(u)}{g''(v)}. \tag{25}$$

By (25) we get $f''(u)/g''(v) = \text{const}$. Then $f''(u) = \text{const}$ and $g''(v) = \text{const}$. Thus $f(u) + g(v)$ is a polynomial of degree 2. By Example 1, our surface is isotropic similar to a subset of (20). $\square$

One can see the same geometrically. The directions of the two parametric curves $u = \text{const}$ and $v = \text{const}$ through each point of a translational surface are conjugate (by the property mentioned in Section 2). The top view of the parametric curves consists of two families of parallel lines. Therefore the top view of the two conjugate directions is the same everywhere. But a pair of conjugate directions and the ratio $a \neq 1$ of isotropic principal curvatures are enough to determine the two isotropic principal directions up to symmetry. By continuity, the top views of the isotropic principal directions are the same everywhere. Consider one parametric curve $v = \text{const}$ and a pair of parametric curves $u = \text{const}$. These three parametric curves intersect at two points. The parametric curves $u = \text{const}$ are translations of each other along the parametric curve $v = \text{const}$. Hence the isotropic normal curvature of the former two parametric curves is the same at these two points. Since the ratio of isotropic principle curvatures is constant, by the (isotropic) Euler formula (see [7, Eq. (4,28)]) it follows that the isotropic principle curvatures are also the same. Thus, again by the Euler formula, the isotropic normal curvature of the parametric curve $v = \text{const}$ is the same everywhere. Thus the latter parametric curve, hence $\alpha(u)$, is a parabolic isotropic circle. Analogously, $\beta(v)$ is a parabolic isotropic circle, and our surface is a paraboloid.

Lemma 25. *Under the assumptions of Theorem 23, if α and β are contained in an isotropic and a non-isotropic plane respectively, then the surface is isotropic similar to a subset of* (21).

Proof Performing an isotropic similarity of the form $z \mapsto z + px + qy$ we can take the plane of β to the plane $z = 0$. After that, performing a rotation about a vertical axis, we can take the plane of α to the plane $x = 0$. Since the surface is admissible, α cannot have vertical tangents. Thus it can be parameterized as $\alpha(u) = (0, -u, f(u))$ for some function $f(u)$. Since the surface $r(u, v) = \alpha(u) + \beta(v)$ is admissible, β cannot have tangents parallel to the y-axis. Thus it can be parameterized as $\beta(v) = (v, g(v), 0)$ for some function $g(v)$. Then the surface is

$$r(u, v) = (v, -u + g(v), f(u)). \tag{26}$$

Hence the isotropic Gaussian and mean curvatures are (see [7, Sections 4–5])

$$K = f'(u)f''(u)g''(v), \quad H = \frac{f'(u)g''(v) + \left(1 + g'(v)^2\right) f''(u)}{2}. \tag{27}$$

Since $K \neq 0$, it follows that $f'(u)f''(u)g''(v) \neq 0$. Therefore the equation $H^2/K = (a+1)^2/(4a)$ is equivalent to

$$\left((1 + g'(v)^2)\frac{f''(u)}{f'(u)} + g''(v)\right)^2 = \frac{(a+1)^2}{a}\frac{f''(u)}{f'(u)}g''(v). \tag{28}$$

By (28) we get $f''(u)/f'(u) = \text{const}$. Hence $f(u) = pe^{\lambda u} + q$ for some constants p, q, λ, where $p, \lambda \neq 0$. Substituting λ for $\frac{f''(u)}{f'(u)}$ in (28), we get (see [49, Section 2.1])

$$g''(v) = \frac{\lambda}{4a}\left(a + 1 \pm \sqrt{(a-1)^2 - 4ag'(v)^2}\right)^2. \tag{29}$$

To solve the resulting ODE, introduce the new variable $p = g'(v)$. Since $g''(v) \neq 0$, it follows that the function $g'(v)$ has a smooth inverse $v(p)$ and there is a well-defined composition $g(p) := g(v(p))$. Substituting $g'(v) = p$ into (29) and using $g''(v) = \frac{dp}{dv} = 1/\frac{dv}{dp} = p/\frac{dg}{dp}$ for $p \neq 0$, we obtain

$$\lambda\frac{dg}{dp} = \frac{4ap}{\left(a + 1 \pm \sqrt{(a-1)^2 - 4ap^2}\right)^2}, \tag{30}$$

$$\lambda\frac{dv}{dp} = \frac{4a}{\left(a + 1 \pm \sqrt{(a-1)^2 - 4ap^2}\right)^2}. \tag{31}$$

Clearly, $a \neq 1$. Integrating, we obtain (see [49, Section 2.2])

$$\lambda g(p) = -\log\left|a + 1 \pm \sqrt{(a-1)^2 - 4ap^2}\right| - \frac{a+1}{a + 1 \pm \sqrt{(a-1)^2 - 4ap^2}} + C_1, \tag{32}$$

$$\lambda v(p) = \frac{(a-1)^2}{4a}\left[\arctan p \mp \arctan\left(\frac{(a+1)p}{\sqrt{(a-1)^2 - 4ap^2}}\right)\right] + \\ + \frac{(a+1)p}{a + 1 \pm \sqrt{(a-1)^2 - 4ap^2}} + C_2 \tag{33}$$

for some constants C_1, C_2. Denote by w the expression in square brackets in (33) plus $\pm\text{sgn}\,(a-1)\cdot\pi/2$. Passing to the new variable w and using the notation $b := (a+1)/(a-1)$, we get (see [49, Section 2.3])

$$\lambda g(w) = \frac{(b^2-1)\log|b - \sin w| + b\sin w + C_1'}{(b^2-1)}, \tag{34}$$

$$\lambda v(w) = \frac{w + b\cos w + C_2'}{(b^2-1)} \tag{35}$$

for some other constants C_1' and C_2'. Performing the isotropic similarity

$$(x, y, z) \mapsto \left(\lambda(b^2-1)x - C_1', \lambda(b^2-1)y - C_2', \frac{z-q}{p}\right)$$

and renaming the parameters u and w to u/λ and v, we bring (26) to form (21). $\square$

Lemma 26. *Under the assumptions of Theorem 23, if α and β are contained in non-isotropic planes, then the surface is isotropic similar to a subset of (22).*

Proof Similarly to the previous lemma, performing an isotropic similarity of the form $z \mapsto px + qy + rz$ and a rotation about a vertical axis we take the planes of α and β to the planes $z = x$ and $z = 0$ respectively. The tangent to α cannot be perpendicular to the x-axis at each point, because otherwise α is a straight line and $a = 0$, contradicting to the assumptions of the theorem. Thus $\alpha'(u) \not\perp Ox$ at some point u. By continuity, the same is true in an interval around u. In what follows switch to an inclusion-maximal interval (u_1, u_2) with this property. Notice that then each endpoint u_k is either an endpoint of the domain of $\alpha(u)$ or there exist finite

$$\lim_{u\to u_k} \alpha(u) \quad \text{and} \quad \lim_{u\to u_k} \alpha'(u)/|\alpha'(u)| \perp Ox.$$

On the interval (u_1, u_2), the curve α can be parameterized as $\alpha(u) = (u, f(u), u)$ for some smooth function $f(u)$. Analogously, on a suitable interval (v_1, v_2) the curve β can be parameterized as $\beta(v) = (v, g(v), 0)$ for some smooth $g(v)$. Therefore a part of our surface can be parameterized as

$$r(u, v) = (u + v, f(u) + g(v), u). \tag{36}$$

Thus the isotropic Gaussian and mean curvatures are (see [7, Sections 4–5] and [43, Eq. (6.3), (6.8)])

$$K = \frac{f''(u)g''(v)}{(f'(u) - g'(v))^4}, \quad H = \frac{(1 + f'(u)^2)g''(v) + (1 + g'(v)^2)f''(u)}{2|f'(u) - g'(v)|^3}. \tag{37}$$

Here $f''(u)g''(v) \neq 0$ and $f'(u) - g'(v) \neq 0$ because $K \neq 0$ and the surface is admissible. Hence the equation $H^2/K = (a + 1)^2/(4a)$ is equivalent to

$$\Big((1 + f'(u)^2)g''(v) + (1 + g'(v)^2)f''(u)\Big)^2 = \frac{(a + 1)^2}{a} f''(u)g''(v)(f'(u) - g'(v))^2. \tag{38}$$

First let us solve the equation in the case when $a = -1$ (this was done in [41, 42]). In this case,

$$f''(u)/(1 + f'(u)^2) = -g''(v)/(1 + g'(v)^2) = c$$

for some constant $c \neq 0$. Hence

$$f(u) = \frac{1}{c} \log|\cos(cu + c_1)| + c_2 \quad \text{and} \quad g(v) = -\frac{1}{c} \log|\cos(cv + c_3)| + c_4$$

for some constants c_1, c_2, c_3, and c_4. Changing the parameters (u, v) to $(u-c_1, v-c_3)/c$ and performing the isotropic similarity $(x, y, z) \mapsto (cx + c_1 + c_3, c(y - c_2 - c_4), cz + c_1)$ we bring (36) to form (22). Notice that there are no points $u_1, u_2 \in \mathbb{R}$ with finite $\lim_{u\to u_k} f(u)$ and $\lim_{u\to u_k} 1/\sqrt{2 + f'(u)^2} = 0$; hence the above maximal intervals (u_1, u_2) and (v_1, v_2) coincide with the domains of $\alpha(u)$ and $\beta(v)$, and (36) actually coincides with the whole given surface.

Now let us prove that for $a \neq -1$ equation (38) has no solutions with $f''(u)g''(v) \neq 0$. The equation is equivalent to

$$(1+f'(u)^2)g''(v)+(1+g'(v)^2)f''(u)\pm(a+1)\sqrt{\frac{f''(u)g''(v)}{a}}(f'(u)-g'(v)) = 0. \tag{39}$$

We may assume that here we have a plus sign and $g''(v) > 0$, otherwise replace $(f(u), g(v))$ by $\operatorname{sgn}(g''(v))(f(\pm u), g(\pm v))$.

If $a = 1$ then (39) is equivalent to

$$\left(\sqrt{f''(u)/g''(v)} + f'(u)\right)^2 + \left(g'(v)\sqrt{f''(u)/g''(v)} - 1\right)^2 = 0.$$

Hence $f'(u) = -1/g'(v)$ is constant. Therefore $f''(u) = 0$, a contradiction.

Assume further $a \neq \pm 1$. For fixed v, a solution $f(u)$ of (39) gives a regular curve in the plane with the coordinates

$$(X, Y) := \left(\sqrt{f''(u)/a}, f'(u)\right)$$

because $f''(u) \neq 0$. By (39), the curve is contained in the conic

$$a(1+g'(v)^2)X^2 + (a+1)\sqrt{g''(v)}XY + g''(v)Y^2 - (a+1)\sqrt{g''(v)}g'(v)X + g''(v) = 0. \tag{40}$$

The conic is irreducible because the determinant of its matrix is

$$-(a-1)^2(g'(v)^2+1)g''(v)^2/4 \neq 0$$

for $a \neq 1$ (see [49, Section 2.4]). Since such irreducible conics (40) for distinct v have a common curve, they actually do not depend on v. Thus the ratio of the coefficients at X and XY is constant. Since $a \neq -1$, it follows that $g'(v) = \text{const}$ and $g''(v) = 0$, a contradiction. Therefore there are no solutions for $a \neq -1$. □

Lemma 27. *There are no admissible translational surfaces with constant nonzero ratio of isotropic principal curvatures formed by an isotropic planar curve α and a nonplanar curve β.*

Proof Assume the converse. Performing a rotation about a vertical axis, we can take the plane of α to the plane $x = 0$. Since the surface is admissible, α cannot have vertical tangents. Thus it can be parameterized as $\alpha(u) = (0, u, f(u))$ for some function $f(u)$. Since the surface $r(u,v) = \alpha(u) + \beta(v)$ is admissible, β cannot have tangents perpendicular to the x-axis. Thus it can be parameterized as $\beta(v) = (v, g(v), h(v))$ for some functions $g(v)$ and $h(v)$. Then the surface is

$$r(u,v) = (v, u + g(v), f(u) + h(v)). \tag{41}$$

Therefore the isotropic Gaussian and mean curvatures are (see [7, Sections 4–5])

$$K = -f''(u)(f'(u)g''(v) - h''(v)), \tag{42}$$

$$H = \frac{f'(u)g''(v) - h''(v) - (1+g'(v)^2)f''(u)}{2}. \tag{43}$$

Here $f''(u) \neq 0$ because $K \neq 0$. Thus the equation $H^2/K = (a+1)^2/(4a)$ is equivalent to

$$\left(1 + g'(v)^2 + \frac{h''(v) - f'(u)g''(v)}{f''(u)}\right)^2 = \frac{(a+1)^2}{a}\,\frac{(h''(v) - f'(u)g''(v))}{f''(u)}. \tag{44}$$

Here the right side (without the factor $(a+1)^2/a$) does not depend on u because equation (44) is quadratic in it with the coefficients not depending on u. Differentiating the right side with respect to u, we get (see [49, Section 2.5])

$$\frac{g''(v)(f'''(u)f'(u) - f''(u)^2) - h''(v)f'''(u)}{f''(u)^2} = 0. \tag{45}$$

If there exists u such that $f'''(u) \neq 0$, then $h''(v) = \text{const} \cdot g''(v)$, otherwise $g''(v) = 0$ identically. In both cases $g'''(v)h''(v) - h'''(v)g''(v) = 0$ identically. Hence β is a planar curve, a contradiction. □

Lemma 28. *There are no admissible translational surfaces with constant nonzero ratio of isotropic principal curvatures formed by a non-isotropic planar curve α and a nonplanar curve β.*

Proof Assume the converse. Performing an isotropic similarity of the form $z \mapsto px + qy + z$ we can take the plane of α to the plane $z = 0$. Performing an appropriate rotation with a vertical axis and restricting to sufficiently small parts of our curves, we may assume that the tangents to α and β are not perpendicular to the x-axis at each point. Then the curves can be parameterized as $\alpha(u) = (u, f(u), 0)$ and $\beta(v) = (v, g(v), h(v))$ for some smooth functions $f(u)$, $g(v)$, and $h(v)$. Therefore a part of our surface can be parameterized as

$$r(u, v) = (u + v, f(u) + g(v), h(v)). \tag{46}$$

Thus the isotropic Gaussian and mean curvatures are (see [7, Sections 4–5] and [43, Eq. (6.3), (6.8)])

$$K = \frac{f''(u)h'(v)\left(g''(v)h'(v) - h''(v)(g'(v) - f'(u))\right)}{(f'(u) - g'(v))^4}, \tag{47}$$

$$H = -\frac{(1 + g'(v)^2)f''(u)h'(v) + (1 + f'(u)^2)\left(g''(v)h'(v) - h''(v)(g'(v) - f'(u))\right)}{2|f'(u) - g'(v)|^3}. \tag{48}$$

Here $f''(u), h'(v), f'(u) - g'(v) \neq 0$ because $K \neq 0$. Hence the equation $H^2/K = (a + 1)^2/(4a)$ is equivalent to

$$a\left((g'(v)^2 + 1)f''(u) + (f'(u)^2 + 1)\left(g''(v) - \frac{h''(v)}{h'(v)}(g'(v) - f'(u))\right)\right)^2 -$$
$$-(a + 1)^2(f'(u) - g'(v))^2 f''(u)\left(g''(v) - \frac{h''(v)}{h'(v)}(g'(v) - f'(u))\right) = 0. \tag{49}$$

Fix a value of v. A solution $f(u)$ of (49) gives a regular curve in the plane with the coordinates $(X, Y) := \left(f''(u), f'(u)\right)$ because $f''(u) \neq 0$. The curve is disjoint with the line $X = 0$. By (49), the curve is contained in the algebraic curve

$$a\left((g'(v)^2 + 1)X + (Y^2 + 1)L(Y)\right)^2 - (a + 1)^2 X(Y - g'(v))^2 L(Y) = 0, \tag{50}$$

where

$$L(Y) := g''(v) - h''(v)(g'(v) - Y)/h'(v).$$

The expression $L(Y)$ is not a zero polynomial in Y, because otherwise (50) reduces to $X = 0$, whereas the above regular curve is disjoint with the line $X = 0$.

Let us prove that the algebraic curve (50) is irreducible (where an irreducible curve of multiplicity two is also viewed as irreducible). Indeed, otherwise the left side equals

$$a(g'(v)^2 + 1)^2(X - P_1(Y))(X - P_2(Y))$$

for some complex polynomials $P_1(Y)$ and $P_2(Y)$. Consider (50) as a quadratic equation in X. Then its discriminant $D(Y) = a^2(g'(v)^2 + 1)^4(P_1(Y) - P_2(Y))^2$ is the square of a polynomial in Y. A direct computation gives (see [49, Section 2.6])

$$D(Y) = (a + 1)^2(Y - g'(v))^2 L(Y)^2 \cdot$$
$$\cdot\left((a - 1)^2 g'(v)^2 - 4a - 2(a + 1)^2 g'(v)Y + \left((a - 1)^2 - 4ag'(v)^2\right)Y^2\right). \tag{51}$$

Assume $a \neq -1$, otherwise the left side of (50) is the square of a linear in X, hence irreducible, polynomial. All factors of $D(Y)$ except the last one are complete squares and not zero polynomials in Y. Hence the last factor, which is at most quadratic in Y, is a square of a polynomial in Y. Hence its discriminant (see [49, Section 2.7]) $16(a - 1)^2 a(g'(v)^2 + 1)^2$ vanishes. Since $a \neq 0$, we get $a = 1$. Then $D(Y) \leq 0$ with the equality only for a finite number of real values of Y. Therefore (50) has only a finite number of real points (X, Y) and cannot contain a regular curve. This contradiction proves that (50) is irreducible.

Since irreducible curves (50) for distinct v contain the same regular curve, they must all coincide. Thus the ratio of the free term and the coefficient at Y in (50) is constant (including the case when one of the coefficients or both vanishes). Hence the ratio of the two coefficients of the linear polynomial $L(Y)$ is constant. Thus $p(h'(v)g''(v) - h''(v)g'(v)) - qh''(v) = 0$ for some constants p and q not vanishing simultaneously. Therefore $((pg'(v) + q)/h'(v))' = 0$, hence $pg'(v) + q = rh'(v)$ and $pg(v) + qv + rh(v) + s = 0$ for some constants r and s. Thus the curve $\beta(v) = (v, g(v), h(v))$ is planar, a contradiction. $\square$

8 Dual-translational surfaces

8.1 Isotropic metric duality

The principle of duality is a crucial concept in projective geometry. For example, in projective 3-space, points are dual to planes and vice versa, straight lines are dual to straight lines, and inclusions are reversed by the duality.

In contrast to Euclidean geometry, isotropic geometry possesses a *metric duality*. It is defined as the polarity with respect to the unit isotropic sphere, which maps a point $P = (p_1, p_2, p_3)$ to the non-isotropic plane P^* with the equation $z = p_1x + p_2y - p_3$, and vice versa. For two points P and Q at isotropic distance d, the dual planes P^* and Q^* intersect at the isotropic angle d. The latter is defined as the difference between the slopes of the two lines obtained in a section of P^* and Q^* by an isotropic plane orthogonal to the line $P^* \cap Q^*$.

The following properties of the metric duality are straightforward. Parallel points, defined as points having the same top view, are dual to parallel planes. Two non-parallel lines in a non-isotropic plane are dual to two non-parallel lines in a non-isotropic plane. Two parallel lines in a non-isotropic plane are dual to two non-parallel lines in an isotropic plane.

The *dual* Φ^* of an admissible surface Φ is the set of points dual to the tangent planes of Φ. If Φ is the graph of a smooth function f, then the tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ is

$$z = xf_x(x_0, y_0) + yf_y(x_0, y_0) - (x_0f_x(x_0, y_0) + y_0f_y(x_0, y_0) - f(x_0, y_0)).$$

Hence Φ^* is parameterized by

$$x^*(x, y) = f_x(x, y), \quad y^*(x, y) = f_y(x, y), \quad z^*(x, y) = xf_x + yf_y - f. \qquad (52)$$

If Φ has parametric form $(x(u, v), y(u, v), z(u, v))$, then Φ^* is parameterized by

$$x^*(u, v) = \frac{y_u z_v - y_v z_u}{x_v y_u - x_u y_v}, \quad y^*(u, v) = \frac{x_u z_v - x_v z_u}{x_u y_v - x_v y_u}, \quad z^*(u, v) = xx^* + yy^* - z. \qquad (53)$$

It is important to note that Φ^* may have singularities that correspond to parabolic points of Φ, where $K = 0$, and doubly-tangent planes. This duality relationship is reflected in the following expressions that relate the isotropic curvatures of dual surfaces, as shown in [60]: $H^* = H/K$ and $K^* = 1/K$.

Thus the dual of an isotropic CRPC surface is again an isotropic CRPC surface because $(H^*)^2/K^* = H^2/K$. The classes of rotational, parabolic rotational, ruled, and helical CRPC surfaces are clearly invariant under the duality. Each surface (3), (7), (8), (10), and (11) is isotropic similar to its dual. Two surfaces (2) are isotropic similar to the duals of each other.

More properties of the metric duality can be found in [8].

8.2 Dual-translational isotropic CRPC surfaces

For the translational surfaces, the duality leads to a new type of surfaces: the ones with a conjugate net of isotropic geodesics. A curve on a surface is an *isotropic geodesic* if its top view is a straight line segment. Two families of curves form a *conjugate net* if any two curves from distinct families intersect and their directions at the intersection point are conjugate (see the definition in Section 2).

Proposition 29. *(See Fig. 7) The dual surfaces of* (21) *and* (22) *are up to general isotropic similarity respectively*

$$r_b^*(u,v) = \exp u \begin{bmatrix} \frac{\cos v}{b-\sin v} \\ 1 \\ \frac{b-b^3+v\cos v}{(b^2-1)(b-\sin v)} - \log|b-\sin v| + u \end{bmatrix}, \tag{54}$$

$$r^*(u,v) = \frac{1}{\tan u + \tan v} \begin{bmatrix} \tan v \\ 1 \\ \log|\frac{\cos v}{\cos u}| - u\tan u + v\tan v \end{bmatrix}. \tag{55}$$

They have a constant ratio (equal to $(b-1)/(b+1)$ *and* -1 *respectively) of isotropic principal curvatures and possess a conjugate net of isotropic geodesics. The domains of maps* (54) *and* (55) *are subsets of the domains of* (21) *and* (22)*, where the maps are injective; see Theorem 23.*

The proposition is proved by direct calculation with the help of (53) (see [49, Section 4]). We still have to show that duals to translational surfaces possess a conjugate net of isotropic geodesics. At all points of a curve $u = \text{const}$ on a translational surface, the tangents to the curves $v = \text{const}$ are parallel and form a general cylinder. Hence all tangent planes along the curve $u = \text{const}$ are parallel to one line. Then the duals of those planes form a section of the dual surface by an isotropic plane, which is an isotropic geodesic. The same is true for the tangent planes along each curve $v = \text{const}$. Since curves $u = \text{const}$ and $v = \text{const}$ form a conjugate net on the translational surface and any projective

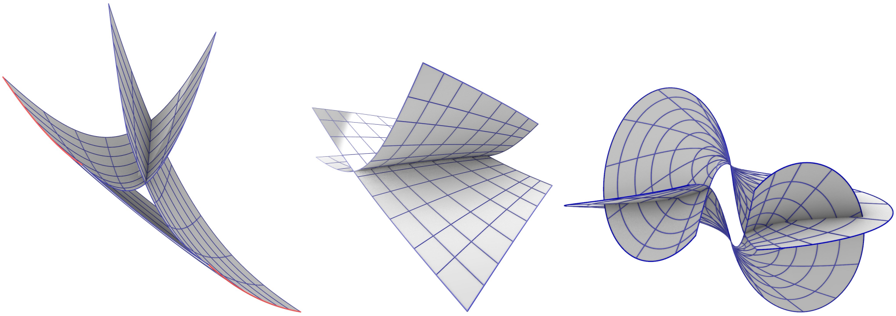

Fig. 7: Duals of the translational isotropic CRPC surfaces (from the left to the right): surfaces (54) for $a > 0$ and $a < 0$; surface (55). The red curves are the singular curves $b \sin v = 1$ of surface (54) with $a > 0$.

duality maps conjugate tangents to conjugate tangents, the metric dual to a translational surface possesses a conjugate net of geodesics.

Remark 30. Surfaces with a conjugate net of Euclidean geodesics have been determined by A. Voss [61]. The conjugate net of geodesics is remarkably preserved by a one-parameter family of isometric deformations. The conjugate nets of geodesics are reciprocal parallel to the asymptotic nets of surfaces with constant negative Gaussian curvature. Discrete versions of Voss nets are quad meshes with planar faces that are flexible when the faces are rigid and the edges act as hinges. We refer the reader to R. Sauer [62]. Analogous properties hold for the isotropic counterparts of Voss surfaces if one defines an isometric deformation in isotropic space as one which preserves the top view and isotropic Gauss curvature [58]. We will report on this and related topics in a separate publication.

9 Open problems

Following the general philosophy discussed in Section 1, one can try to apply the methods developed for the classification of helical and translational isotropic CRPC surfaces in Sections 6–7 to their analogs in Euclidean geometry; cf. [20]. The case of translational surfaces generated by two spatial curves remains open in both geometries.

It is natural to extend the search for CRPC surfaces to other Cayley–Klein geometries such as Galilean or Minkowski geometry, and Cayley–Klein vector spaces [63, 64]. The transition from Euclidean to pseudo-Euclidean (Minkowski) geometry is not expected to lead to significant differences, but more case distinctions. It is relative differential geometry with respect to a hyperboloid. We may return to this topic in future research if it appears to be rewarding.

A The singularities in the top view

For the classification of ruled CRPC surfaces (see Section 5), we need the following two well-known lemmas from singularity theory, which we could not find a good reference for.

Lemma 31. *If R_t is an analytic family of lines in the plane, then for all t in some interval I the lines R_t satisfy one of the following conditions*

(i) they have a common point;
(ii) they are parallel;
*(iii) they are tangent to one regular curve (*envelope*) forming a part of the boundary of the union $\bigcup_{t\in I} R_t$.*

Proof Assume without loss of generality that all R_t are not parallel to the y axis for t in some interval I_1. Let $L(x,y,t) := y + a(t)x + b(t) = 0$ be the equation of the line R_t. For each $n = 1, 2, \ldots$ consider the subset Σ_n of $\mathbb{R}^2 \times I_1$ given by $L(x,y,t) = \frac{\partial}{\partial t} L(x,y,t) = \cdots = \frac{\partial^n}{\partial t^n} L(x,y,t) = 0$ but $\frac{\partial^{n+1}}{\partial t^{n+1}} L(x,y,t) \neq 0$, and also the subset Σ_∞ given by $\frac{\partial^n}{\partial t^n} L(x,y,t) = 0$ for all $n = 0, 1, \ldots$. By the analyticity, we may restrict to a subsegment $I_2 \subset I_1$ such that no connected component of Σ_n is contained in a plane of the form $\mathbb{R}^2 \times \{t\}$. Consider the following 3 cases.

Case (i): $\Sigma_\infty \neq \emptyset$. Then take a point $(x,y,t) \in \Sigma_\infty$. By the analyticity $L(x,y,t) = 0$ for all t. Hence (x,y) is a point common to all R_t.

Case (ii): $\Sigma_\infty = \Sigma_n = \emptyset$ for all n. Then the system $L(x,y,t) = \frac{\partial}{\partial t} L(x,y,t) = 0$ has no solutions, i.e., $a'(t) = 0$ and $b'(t) \neq 0$ everywhere. Hence $a(t) = \text{const}$, $b(t)$ is monotone, and all R_t are parallel.

Case (iii): $\Sigma_n \neq \emptyset$ for some n. Then take a point $(x,y,t) \in \Sigma_n$ and consider the map $G(x,y,t) := \left(L(x,y,t), \frac{\partial^n}{\partial t^n} L(x,y,t)\right)$; this generalizes the argument from [50, Section 5.21], where $n = 1$. Let us show that the top view (projection to the xy-plane along the t-axis) of $G^{-1}(0,0)$ is the required curve (envelope). Since $\frac{\partial L}{\partial t} = 0$, $\frac{\partial^{n+1} L}{\partial t^{n+1}} \neq 0$, and $\frac{\partial L}{\partial y} \neq 0$, it follows that the differential dG is surjective. Then by the Implicit Function Theorem, the intersection of $G^{-1}(0,0)$ with a neighborhood of (x,y,t) is a regular analytic curve with the tangential direction (dx, dy, dt) given by

$$\frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial x} dx + \frac{\partial L}{\partial y} dy = \frac{\partial^{n+1} L}{\partial t^{n+1}} dt + \frac{\partial^{n+1} L}{\partial t^n \partial x} dx + \frac{\partial^{n+1} L}{\partial t^n \partial y} dy = 0;$$

cf. [50, Proof of Proposition 5.25]. Since $\frac{\partial L}{\partial t} = 0$ and $\frac{\partial^{n+1} L}{\partial t^{n+1}} \neq 0$, it follows that the top view of the curve is a regular analytic curve tangent to L_t. Since no component of Σ_n is contained in the plane $\mathbb{R}^2 \times \{t\}$, it follows that the top view is tangent to L_t for each t in a sufficiently small interval $I_3 \subset I_2$.

The resulting envelope cannot be a straight line, otherwise, we have case (i). Then it has a non-inflection point. Then for a sufficiently small $I \subset I_3$, a part of the curve is contained in the boundary of the union $\bigcup_{t\in I} R_t$. $\square$

Lemma 32. *Assume that an analytic curve in $\mathbb{R}^3$ has a vertical tangent at a point O and does not coincide with the tangent. Then the limit of the top view of the tangent at a point P tending to O coincides with the top view of the limit of the osculating plane at a point P tending to O. In particular, both limits exist.*

Proof Let $r(t)$ be the arclength parametrization of the curve with $r(0) = O$. Since the curve is not a vertical line, $r'(t) \neq$ const. Hence by the analyticity $r'(t) = r'(0)+t^n a(t)$ for some integer $n \geq 1$ and a real analytic vector-function $a(t)$ such that $a(0) \neq 0$. Since $|r'(t)| =$ const, it follows that $a(0) \perp r'(0)$, i.e. $a(0)$ is horizontal.

The top view of the tangent at $P = r(t)$, where $t \neq 0$ is small enough, is parallel to the top view of $a(t)$ because $r'(0)$ is vertical. Hence the limit of the former is parallel to $a(0)$.

The osculating plane at $P = r(t)$, where $t \neq 0$ is small enough, is parallel to the linear span of $r'(t) = r'(0) + t^n a(t)$ and $r''(t)/t^{n-1} = na(t) + ta'(t)$. As $t \to 0$, the latter tends to the span of $r'(0)$ and $na(0)$, with the top view again parallel to $a(0)$. $\square$

An exciting study of simply isotropic minimal surfaces containing isotropic lines can be found in the recent work by Akamine and Fujino [65], which may provide additional insights related to the concepts discussed above.

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