Exact Parameterization of Convolution Surfaces and Rational Surfaces with Linear Normals

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Abstract

It is shown that curves and surfaces with a linear field of normal vectors are dual to graphs of univariate and bivariate polynomials. We discuss the geometric properties of these curves and surfaces. In particular, it is shown that the convolution with general rational curves and surfaces yields again rational curves and surfaces.

Key words: Rational curves and surfaces, linear normals, dual representation, convolution, offsetting.

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1 Introduction

The notions of *convolution surfaces* and *Minkowski sums* in two and three dimensions are used in various fields of mathematics, e.g., mathematical morphology, computer graphics, convex geometry and computational geometry, and there is a close connection between them. Roughly speaking, the untrimmed boundary curve or surface of the Minkowski sum is the convolution curve/surface of the boundaries.

In the curve case, various algorithms for computing Minkowski sums exist (Kaul and Farouki, 1995; Kohler and Spreng, 1995; Lee, Kim and Elber, 1998a,b; Ramkumar, 1996; Farouki, 2003). The main issue is to trim away those parts of the convolution curve that do not contribute to the outer boundary of the Minkowski sum.

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Another important problem consists in finding an exact description of the convolution. Though the set of algebraic curves and surfaces is closed under convolution, this result is of little practical value, since the resulting degrees are far too high to be useful. Also, one is often interested in curves and surfaces that admit a rational parametric representation, since they can easily be fed into standard CAD systems.

As an important special case, offset surfaces (convolutions with spheres) have thoroughly been discussed, where certain rational surfaces are equipped with rational offset surfaces. For instance, this is true for surfaces which degenerate to space curves, and for quadrics (Landsmann, Schicho and Winkler, 2001; Lü, 1994; Peternell and Pottmann, 1998; Schicho, 2000).

Rational Convolution surfaces of more general surfaces did not receive much attention so far. Recently, Pottmann and Mühltaler (2003) have analyzed the case of two ruled surfaces, and convolutions between paraboloids and general rational surfaces were analyzed by Peternell and Manhart (2003).

In this paper, we generalize the latter surface to the case of convolutions between surfaces with linear normals (LN surfaces) and general rational surfaces. LN surfaces, which were studied in (Jüttler and Sampoli, 2000) have sufficient flexibility to model smooth surfaces without parabolic points. Moreover, we will show that their convolution surfaces with general parametric surfaces have explicit parametric representation, which are even rational for rational surfaces.

This paper is organized as follows. The first three sections are devoted to LN surfaces, their dual representation, and the available constructions. Then we discuss the so–called relative differential geometry of these surfaces. Sections 6 and 7 discuss Minkowski sums, convolution surfaces, and the parameterization of convolution surfaces. Finally, we conclude this paper.

2 Preliminaries

This paper is devoted to a special class of rational surfaces.

Definition 1 *Consider a polynomial (or, more general, a rational) surface* $\mathbf{p}(u,v)$*. This surface is said to be an* **LN** *surface, if its normal vectors admit a linear representation of the form*

$$\vec{\mathbf{N}}(u,v) = \vec{\mathbf{a}}u + \vec{\mathbf{b}}v + \vec{\mathbf{c}} \tag{1}$$

with certain constant coefficient vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}} \in \mathbb{R}^3$*. More precisely, it satisfies* *the equations*

$$\mathbf{p}_u(u,v)\cdot\vec{\mathbf{N}}(u,v) \equiv \mathbf{p}_v(u,v)\cdot\vec{\mathbf{N}}(u,v) \equiv 0, \tag{2}$$

where $\mathbf{p}_u(u,v) = (\partial/\partial u)\mathbf{p}(u,v)$, $\mathbf{p}_v(u,v) = (\partial/\partial v)\mathbf{p}(u,v)$.

The equations (2) can be seen as linear constraints on the space of polynomial or rational parametric surfaces, and this approach has been used by Jüttler and Sampoli (2000) for generating LN surface patches matching given Hermite boundary data. In this paper, we will study the geometrical properties by using the so–called dual representation of these surfaces, where the surface is seen as the envelope of its tangent planes.

Remark 2 (1) If the three vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are linearly dependent, then the surface $\mathbf{p}(u,v)$ describes a general cylinder, since the unit normals $\vec{\mathbf{N}}/||\vec{\mathbf{N}}||$ are contained in a great circle on the unit sphere.

(2) In the remainder of this paper we assume that the three vectors are linearly independent. Without loss of generality we may then assume that

$$\vec{\mathbf{a}} = (1,0,0)^\top,\ \vec{\mathbf{b}} = (0,1,0)^\top,\ \vec{\mathbf{c}} = (0,0,1)^\top, \tag{3}$$

i.e., $\vec{\mathbf{N}}(u,v) = (u,v,1)^\top$. This situation can be achieved by a uniform scaling of $\mathbb{R}^3$, a suitable choice of Cartesian coordinates, and a linear parameter transformation $u = u(u',v'), v = v(u',v')$.

Proposition 3 *Under the assumptions of Remark 2, the tangent planes of an LN surface have the equations*

$$T(u,v) : f(u,v) + u\,x + v\,y + z = 0, \tag{4}$$

where $f(u,v) = -\mathbf{p}(u,v)\cdot\vec{\mathbf{N}}(u,v)$ is a polynomial or rational function, in the case of a polynomial or rational LN surface, respectively. On the other hand, given a system of tangent planes of the form (4) *with a polynomial or rational function $f(u,v)$, the envelope surface*

$$\mathbf{p}(u,v) = (-f_u, -f_v, -f + uf_u + vf_v)^\top \tag{5}$$

is a polynomial or rational LN surface.

Proof. The envelope surface $\mathbf{p} = (x,y,z)$ satisfies the equations

$$\begin{aligned} T(u,v) : f(u,v) + ux + vy + z &= 0,\\ T_u(u,v) : f_u(u,v) + x &= 0,\\ T_v(u,v) : f_v(u,v) + y &= 0, \end{aligned} \tag{6}$$

and the normal vector evaluates to

$$\vec{\mathbf{N}}(u,v) = (f_{uu}f_{vv} - f_{uv}^2)(u,v,1)^\top. \tag{7}$$

Remark 4 Due to (2), singular points of the envelope surface (5) are characterized by $f_{uu}f_{vv} - f_{uv}^2 = 0$. In addition, the Gaussian curvature of the envelope equals

$$K(u,v) = \frac{1}{(f_{uu}f_{vv} - f_{uv}^2)(1+u^2+v^2)^2}. \tag{8}$$

Consequently, the algebraic curve $f_{uu}f_{vv} - f_{uv}^2 = 0$ in the (u,v)–parameter domain separates elliptic $(K > 0)$ and hyperbolic $(K < 0)$ points on the LN surface.

3 The dual representation

There exist several interesting relations between the LN-surfaces $\mathbf{p}(u,v)$ defined by a polynomial or rational function f and the associated graph surface

$$\mathbf{q}(u,v) = (u, v, f(u,v))^\top, \tag{9}$$

since the graph surface is dual to the LN surface in the sense of projective geometry.

The points of $\mathbf{q}(u,v)$ are elliptic, parabolic or hyperbolic, if the sign of

$$\det H(f) = f_{uu}f_{vv} - f_{uv}^2 \tag{10}$$

is $1, 0$ or -1, respectively. Clearly, the parabolic points either form an algebraic curve, or the entire graph surface consists of parabolic points only. In the latter case, $\mathbf{q}(u,v)$ is a general cylinder surface.

Corollary 5 *Elliptic and hyperbolic points of the graph surface* $\mathbf{q}(u,v)$ *correspond to elliptic and hyperbolic points of the LN surface* $\mathbf{p}(u,v)$*. Parabolic points of the graph surface* $\mathbf{q}(u,v)$ *correspond to singular points of* $\mathbf{p}(u,v)$*.*

Proof. These facts are consequences of (7), (8) and (10).

Remark 6 Graph surfaces $\mathbf{q}(u,v)$, which are *general cylinders*, correspond to singular surfaces $\mathbf{p}(u,v)$, which degenerate into planar curves. More precisely, the function f can be assumed to take the form

$$f(u,v) = d\,u + g(v) \tag{11}$$

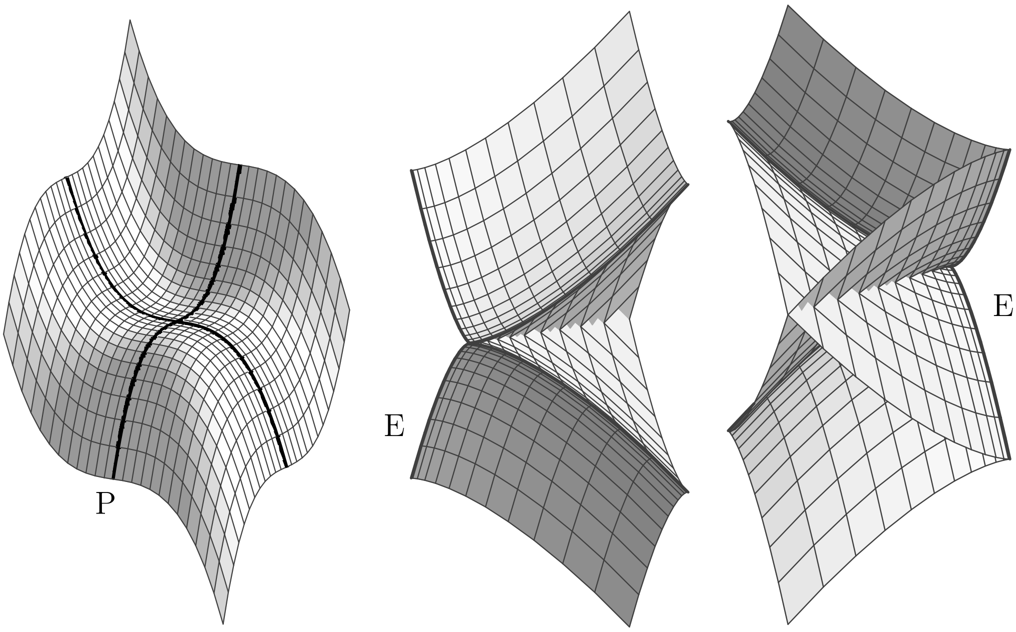


Fig. 1. Graph surface (left) of a cubic polynomial and the associated LN surface (center and right).

with a real constant d and a rational function $g(v)$. The envelope surface (5) degenerates into the planar curve

$$(-d,\, -g'(v),\, g(v) + vg'(v))^\top. \tag{12}$$

If the envelope surface has a self–intersection (i.e., a *double line*), then its points correspond to pairs of points of $\mathbf{q}(u, v)$ with coinciding tangent planes. Consequently, if f is a convex function, then the envelope does not have any self–intersections.

We illustrate these observations by a first example of an LN surface, see Figure 1. The function f is equal to $u^3 - v^3$, and the LN surface has the parametric representation

$$\mathbf{p}(u, v) = (-3u^2, 3v^2, 2u^3 - 2v^3)^\top \tag{13}$$

The parabolic lines (marked with P) on the graph surface are $u = 0$ and $v = 0$. The associated LN surface has 2 edges of regression (E), which intersect in the point $(0, 0, 0)$. Each of them is a planar cubic curve with a cusp (i.e., equivalent to Neil's parabola). In addition, it has a double line, which corresponds to the double tangent planes along the curve $u = v$, since the tangent planes at (u, u) and $(-u, -u)$ are identical.

4 Construction of LN surfaces

We summarize two constructions of LN surfaces. For both of them, the input consists of three points $\mathbf{v}_i \in \mathbb{R}^3$ with associated normal vectors $\vec{\mathbf{n}}_i \in \mathbb{R}^3$, $(i = 0, 1, 2)$. The normal vectors are not assumed to be normalized.

4.1 The problem

Both constructions generate a triangular surface patch $\mathbf{p}(u, v)$, whose parameter domain is a triangle $\triangle \subset \mathbb{R}^2$ with vertices $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$. The parameter pairs $(u, v) \in \triangle$ are described by their barycentric coordinates (r, s, t) with respect to the domain triangle, i.e.

$$(u, v) = r\mathbf{w}_1 + s\mathbf{w}_2 + t\mathbf{w}_3, \quad \text{satisfying} \quad r + s + t = 1 \tag{14}$$

where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{R}^2$ are the vertices of $\triangle$.

The patch $\mathbf{p}(u, v)$ is either a triangular Bézier patch (cf. Farin, Hoschek and Kim, 2002) or a collection of such patches, which interpolates the given three points, i.e.,

$$\mathbf{p}(\mathbf{w}_i) = \mathbf{v}_i. \tag{15}$$

In addition, in order to produce a patch of an LN surface, the normal at a point $\mathbf{p}(u, v)$ is to be parallel to

$$\vec{\mathbf{N}}(r, s, t) = r\mathbf{n}_1 + s\mathbf{n}_2 + t\mathbf{n}_3, \tag{16}$$

where (r, s, t) are the barycentric coordinates of (u, v), cf. (14). This implies the conditions

$$\frac{\partial}{\partial u}\mathbf{p}\bigg|_{(u,v)=r\mathbf{w}_1+s\mathbf{w}_2+t\mathbf{w}_3} \cdot \vec{\mathbf{N}}(r, s, t) = \frac{\partial}{\partial v}\mathbf{p}\bigg|_{(u,v)=r\mathbf{w}_1+s\mathbf{w}_2+t\mathbf{w}_3} \cdot \vec{\mathbf{N}}(r, s, t) = 0 \tag{17}$$

4.2 Two constructions

Both constructions consists of two steps.

(1) *Construction of boundary curves.* For any pair of points $\mathbf{v}_i$, $\mathbf{v}_j$, $i < j$, we construct a polynomial boundary curve $\mathbf{x}_{i,j}(t)$, $t \in [0, 1]$ of the triangular surface patch. In order to obtain patches which can be joined to form a globally G^1 surface, the boundaries should be fully determined by the points $\mathbf{v}_i$, $\mathbf{v}_j$ and vertex normals $\vec{\mathbf{n}}_i$, $\vec{\mathbf{n}}_j$.

The boundary $\mathbf{x}_{i,j}$ is to satisfy

$$\mathbf{x}_{i,j}(0) = \mathbf{v}_i, \quad \mathbf{x}_{i,j}(1) = \mathbf{v}_j, \quad \mathbf{x}'_{i,j}(t) \cdot [(1-t)\vec{\mathbf{n}}_i + t\vec{\mathbf{n}}_j] \equiv 0, \tag{18}$$

where $\mathbf{x}' = (\mathrm{d}/\mathrm{d}t)\mathbf{x}$. These conditions lead to linear equations for the coefficients of the polynomial curve, which are solvable, provided that the degree is sufficiently high. The remaining degrees of freedom are used to minimize a suitable energy functional, such as $\int_0^1 (\mathbf{x}''_{ij})^2 \mathrm{d}t$.

(2) *Filling in a patch.* In the second step, we generate a triangular surface patch whose boundary curves are given by $\mathbf{x}_{i,j}(t)$, and satisfy (17); this leads to a system of linear equations.

It turns out that it is generally not possible to fill in a single patch, due to compatibility conditions at the vertices (similar to the vertex enclosure problem). Two solutions to this problem exist:

(a) One may use a single patch with singular points at the vertices. This has to be taken into account already during the construction of the boundaries, which should then satisfy

$$\mathbf{x}'_{i,j}(0) = \mathbf{x}'_{i,j}(1) = \vec{\mathbf{0}} \tag{19}$$

in addition to (18). This approach leads to patches of degree 6. See (Jüttler, 1998) for details.

(b) Alternatively, in order to avoid potential problems with singular points, one may apply the Clough–Tocher split, by filling in a surface patch composed of three triangular surface patches. This leads to three patches of degree 4. This technique is described in (Jüttler and Sampoli, 2000).

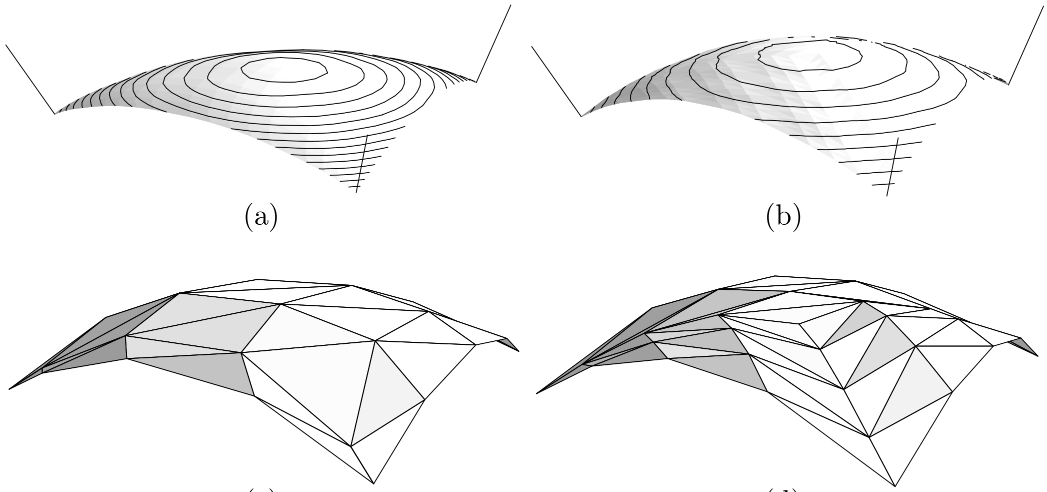
Both approaches lead to systems of linear equations, and the remaining degrees of freedom can be used to minimize suitable fairness measures.

Two examples are shown in Figure 2.

Note that both constructions may produce surfaces which have sharp edges (singular curves), since the prescribed normal field limits the shape of the surface. According to our experience, the surface behaves nicely for boundary data which have been taken from an existing surface without parabolic points, provided that the distances between the sampled points are sufficiently small. This could even be proved for the boundary curves generated in the second construction (Jüttler and Sampoli, 2000).

5 Relations between LN-surfaces and the unit paraboloid

In this section we point to some properties of LN-surfaces in connection with paraboloids. It will turn out that LN-surfaces are in some sense generalizations



(a) (b) (c) (d)

Fig. 2. LN surfaces interpolating three points with associated normal vectors (a,b) and their control nets (c,d). The surfaces have been generated using singularly parameterized surfaces (a,c) and Clough–Tocher splits (b,d).

of paraboloids. This property applies also to the computation of convolution surfaces in section 7.

We recall the parameterization $\mathbf{p}(u,v) = (-f_u, -f_v, -f + uf_u + vf_v)^\top$ of an LN-surface Φ and that its normal vectors are given by $\vec{\mathbf{N}}(u,v) = (u,v,1)^\top$. Additionally we consider the paraboloid Q, represented by

$$\mathbf{q}(u,v) = (u, v, \frac{1}{2}(1 - u^2 - v^2)) = (u, v, q(u,v))^\top. \tag{20}$$

Up to a normalization, Q's normal vectors

$$\vec{\mathbf{N}}_q(u,v) = \frac{1}{\sqrt{1+u^2+v^2}}(u,v,1)^\top$$

agree with those of Φ. Obviously this implies that Φ's tangent planes are parallel to those of Q.

Two points $\mathbf{p}$ of Φ and $\mathbf{q}$ of Q are called *corresponding*, if their normal vectors $\vec{\mathbf{N}}$ and $\vec{\mathbf{N}}_q$ are parallel. Thus, this correspondence is realized by equal (surface) parameters u, v. In section 7, this correspondence applies to the construction of convolution surfaces.

Euclidean differential geometry investigates the unit normal vectors of a surface considered as parameterization of the unit sphere S^2. The *shape operator* or *Weingarten mapping* $w : \mathbf{p}_u \mapsto -\vec{\mathbf{N}}_u, \mathbf{p}_v \mapsto -\vec{\mathbf{N}}_v$ is the differential of the mapping $\mathbf{p}(u,v) \longrightarrow -\vec{\mathbf{N}}$. At each point $\mathbf{p}(u,v)$, w is a linear mapping in the tangent plane. The eigenvalues and eigenvectors of w are the *principal curvatures* and *principal curvature directions* of Φ at $\mathbf{p}$.

By substituting S^2 by the 'unit' paraboloid Q, or in other words, according to the normalization of $\vec{\mathbf{N}}$ by

$$\widetilde{\vec{\mathbf{N}}}(u,v) = (u,v,q(u,v)) = \mathbf{q}(u,v) = (u,v,\frac{1}{2}(1-u^2-v^2)), \tag{21}$$

$\widetilde{\vec{\mathbf{N}}}$ is considered as *relative normalization* with respect to Q. Expressing the second fundamental form of Φ with respect to $\widetilde{\vec{\mathbf{N}}}$, one obtains

$$\widetilde{H}^{-1} = \begin{bmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{bmatrix}, \text{ and } \quad \widetilde{H} = \frac{1}{f_{uu}f_{vv} - f_{uv}^2} \begin{bmatrix} f_{vv} & -f_{uv} \\ -f_{uv} & f_{uu} \end{bmatrix}.$$

$\widetilde{H}$ is the coordinate matrix of the *relative shape operator*

$$\begin{aligned} \widetilde{w} : \mathbf{p}_u &= (-f_{uu}, -f_{uv}, uf_{uu} + vf_{uv}) \mapsto -\widetilde{\vec{\mathbf{N}}}_u = -(1,0,-u) \\ \mathbf{p}_v &= (-f_{uv}, -f_{vv}, uf_{uv} + vf_{vv}) \mapsto -\widetilde{\vec{\mathbf{N}}}_v = -(0,1,-v), \end{aligned}$$

with respect to $\widetilde{\vec{\mathbf{N}}}$. Analogously to the Euclidean case, the eigenvalues and eigenvectors of $\widetilde{H}$ are *principal curvatures* and *principal curvature directions*, with respect to the operator $\widetilde{w}$. Since Q is strongly convex, the eigenvalues and eigenvectors are always real.

6 Convolution surfaces and Minkowski sums

We give a short introduction to the Minkowski sums of two point sets and to the convolution surface of two surfaces.

Given two sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^d$, the *Minkowski sum* of these sets is defined as

$$\mathcal{A} \oplus \mathcal{B} = \{\mathbf{a} + \mathbf{b}, \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{b} \in \mathcal{B}\},$$

see Figure 3 for an example. In particular, algorithms for computing the Minkowski sum of closed (convex) polygons in the plane and polyhedral objects in space have been studied in computational geometry, see Bajaj and Kim (1989); Ramkumar (1996); Kohler and Spreng (1995). Applications include motion planning for polygonal objects in the presence of polygonal obstacles.

Later, these concepts have been generalized to arbitrary shapes in the plane and in space, see Lee, Kim and Elber (1998a,b); Kaul and Farouki (1995); Pottmann and Mühltaler (2003); Peternell and Manhart (2003), where the

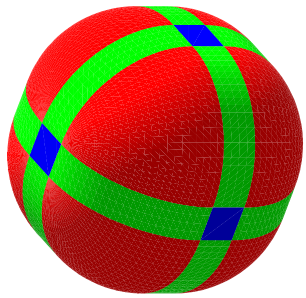


Fig. 3. Minkowski sum of a ball and a cube. The boundary consists of segments of spheres and cylinders and planar patches.

notion of the *convolution* of two (not necessarily convex) objects has been introduced [2].

We consider two regular surfaces A and B in three–dimensional space, which are given by parametric representations $\mathbf{a}(u,v)$ and $\mathbf{b}(s,t)$ with parameter domains $(u,v) \in \Omega_A \subseteq \mathbb{R}^2$ and $(s,t) \in \Omega_B \subset \mathbb{R}^2$, respectively.

Definition 7 *The convolution surface of two surfaces A and B is the set of points*

$$A \star B = \{\, \mathbf{a} + \mathbf{b} \,|\, \mathbf{a} \in A, \mathbf{b} \in B \text{ and } \vec{\mathbf{M}}(\mathbf{a}) \parallel \vec{\mathbf{N}}(\mathbf{b}) \,\},$$

where $\vec{\mathbf{M}}(\mathbf{a})$ and $\vec{\mathbf{N}}(\mathbf{b})$ are the normal vectors of A and B at the points $\mathbf{a} \in A$ and $\mathbf{b} \in B$.

The sum of the coordinate vectors is computed only for those points whose normal vectors are parallel. The definition requires differentiability and regularity of the input surfaces A and B, since otherwise normal vectors do not exist. A more general definition – which is beyond the scope of this paper – could be given by considering 'completed' normal fields.

While Definition 7 uses normal vectors, the convolution surface $A \star B$ is invariant under affine transformations of the objects A and B. This is due to the fact that affine mappings preserve the parallelism of the tangent planes.

Note that there is a close relationship between convolution surfaces and Minkowski sums: the boundary of the Minkowski sum of two sets $\mathcal{A}$, $\mathcal{B}$ is contained in convolution surface of the two boundary surfaces,

$$\partial(\mathcal{A} \oplus \mathcal{B}) \subseteq (\partial\mathcal{A}) \star (\partial\mathcal{B}) \tag{22}$$

[2] This notion should not be confused with the convolution of two functions f and g, which represents roughly spoken, the overlap of f and a reversed and translated version of g.

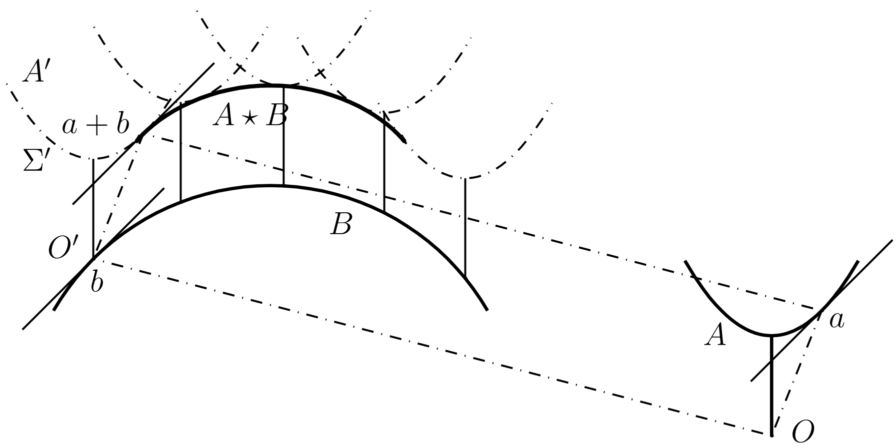


Fig. 4. Kinematic generation convolutions in the curve case.

The convolution $A \star B$ admits the following *kinematic interpretation*[3]. Consider the surface A together with the origin O as a moving system Σ' and let B be fixed, and let A' and O' denote the different positions of A and O. The system Σ' is moved translatory (without any rotational part) such that the point O' travels on the second surface B. The convolution $A \star B$ is generated as the envelope of A' under this two–parameter translational motion. The curve case is visualized in Figure 4.

In particular, if the surface A is a sphere with radius d, centered at O, then the convolution surface $A \star B$ becomes the (untrimmed) offset surface of B at distance d.

7 Parameterization of convolution surfaces

After discussing the general case, we compute convolution surfaces of general rational surfaces and LN surfaces.

7.1 Computation of convolution surfaces

Consider again two surfaces A and B, which are given by parametric representations $\mathbf{a}(u, v)$ and $\mathbf{b}(s, t)$ with parameter domains Ω_A, Ω_B. Let $\vec{\mathbf{M}}(u, v)$ and $\vec{\mathbf{N}}(s, t)$ be their normal vectors, and

$$\mathbf{M}_0(u, v) = \frac{\mathbf{M}(u, v)}{||\mathbf{M}(u, v)||}, \quad \mathbf{N}_0(s, t) = \frac{\mathbf{N}(s, t)}{||\mathbf{N}(s, t)||} \tag{23}$$

[3] A slightly different kinematic generation of $A \star B$ has been discussed by Pottmann and Mühltaler (2003).

the corresponding unit normal vectors. In order to find the convolution surface, we have to construct a reparameterization

$$\phi : \Omega_B^* \to \Omega_A : (s,t) \mapsto (u(s,t), v(s,t)) \tag{24}$$

which is defined for a certain subset $\Omega_B^* \subseteq \Omega_B$, such that the normal vectors $\vec{\mathbf{M}}(u(s,t), v(s,t))$ and $\vec{\mathbf{N}}(s,t)$ at $\mathbf{a}$ and $\mathbf{b}$ are parallel.

The set Ω_B^* should be chosen as the maximal subset of Ω_B, such that either the Gaussian image $\vec{\mathbf{N}}(\Omega_B^*)$ of B or its reflected version $-\vec{\mathbf{N}}(\Omega_B^*)$ is contained in the Gaussian image $\vec{\mathbf{M}}_0(\Omega_A)$ of A. In addition, we assume that the unit normals of the first surface $\vec{\mathbf{M}}_0(u,v)$ define a bijective mapping $\Omega_A \to \vec{\mathbf{N}}_0(\Omega_A)$, and $\vec{\mathbf{N}}_0(\Omega_A)$ is contained in an open hemisphere of the unit sphere [4]. Under these assumptions, the reparameterization exists and it is unique.

Then,

$$\mathbf{c} = \mathbf{a}(u(s,t), v(s,t)) + \mathbf{b}(s,t). \tag{25}$$

is a parametric representation of the convolution surface of $A^* = \mathbf{a}(\phi(\Omega_B^*))$ and $B^* = \mathbf{b}(\Omega_B^*)$. For general rational surfaces A and B, this reparameterization cannot be written down explicitly.

7.2 Convolution of LN surfaces and rational surfaces

In this section we want to investigate parameterizations of the convolution $A \star B$ of an LN–surface A and a rational surface B. We may assume that the coordinate system has been chosen such that the LN–surface A is given by a parameterization

$$\mathbf{a}(u,v) = (-f_u, -f_v, -f + uf_u + vf_v).$$

As observed earlier in section 2, the normal vector $\vec{\mathbf{M}}$ of A is proportional to $\vec{\mathbf{M}}(u,v) = (u,v,1)$ at regular points (which are characterized by $f_{uu}f_{vv} - f_{uv} \neq 0$). In this case, the unit normals $\vec{\mathbf{M}}_0(u,v)$ are obtained in the upper hemisphere.

For the sake of simplicity, we choose $\Omega_A = \mathbb{R}^2$ throughout this section. The second surface B is assumed to admit a smooth local parameterization

$$\mathbf{b} : (s,t) \in G \subset \mathbb{R}^2 \to \mathbb{R}^3.$$

[4] This is the case if and only if there exists a vector $\vec{\mathbf{z}}_0$, such that $\vec{\mathbf{N}}_0(u,v) \cdot \vec{\mathbf{z}}_0 > 0$ holds for all $(u,v) \in \Omega_A$.

Two points $\mathbf{a} \in A$ and $\mathbf{b} \in B$ correspond to each other, if the normal vectors $\vec{\mathbf{M}}$ and $\vec{\mathbf{N}}$ at $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent,

$$\vec{\mathbf{M}}(\mathbf{a}) = \lambda \vec{\mathbf{N}}(\mathbf{b}), \ \lambda \neq 0. \tag{26}$$

Then, $\mathbf{a} + \mathbf{b}$ is a point of the convolution surface $C = A \star B$.

Using the normal vector $\vec{\mathbf{N}}(s,t) = (n_1(s,t), n_2(s,t), n_3(s,t))$ of B, the condition (26) can be rewritten as

$$(u, v, 1) = \lambda(n_1, n_2, n_3)(s,t). \tag{27}$$

which implies

$$u(s,t) = \frac{n_1(s,t)}{n_3(s,t)} \quad \text{and} \quad v(s,t) = \frac{n_2(s,t)}{n_3(s,t)} \tag{28}$$

provided that $n_3(s,t) \neq 0$. The latter condition is satisfied, since the Gaussian image $\vec{\mathbf{N}}_0(\Omega_B^*)$ is assumed to be contained in $\vec{\mathbf{M}}_0(\Omega_A)$

The parametric representation $\mathbf{c}(s,t)$ of the convolution $C = A \star B$ is now obtained by applying the reparameterization (28) to A and evaluating the sum

$$\mathbf{c}(s,t) = \mathbf{a}(\frac{n_1(s,t)}{n_3(s,t)}, \frac{n_2(s,t)}{n_3(s,t)}) + \mathbf{b}(s,t).$$

If B is a rational surface, the reparameterization $\phi : (s,t) \to (u,v)$ is a rational mapping and the convolution $C = A \star B$ is a rational surface.

Theorem 8 *The convolution surface $A \star B$ of an LN–surface A and a parameterized surface B has an explicit parametric representation. If A and B are rational surfaces, their convolution $A \star B$ is rational, too.*

An example is shown in Figure 5, where we visualize the convolution surface of a quadratic triangular patch with an LN surface of degree 6.

Remark 9 The reparameterization ϕ is regular if and only if the determinant of the Jacobian $J\phi$ does not vanish. This determinant evaluates to

$$\det(J\phi) = \frac{1}{n_3^3} \det(\vec{\mathbf{N}}, \vec{\mathbf{N}}_s, \vec{\mathbf{N}}_t). \tag{29}$$

After some computations one arrives at

$$\det(J\phi) = \frac{1}{n_3^3} \det(\vec{\mathbf{N}}, \vec{\mathbf{N}}_s, \vec{\mathbf{N}}_t) = \frac{1}{n_3^3} \det(G_b)^2 k_b, \tag{30}$$

where G_b is the first fundamental form of B, and k_b its Gaussian curvature. We mention two special cases which correspond to a singular Jacobian (29) of the reparameterization ϕ:

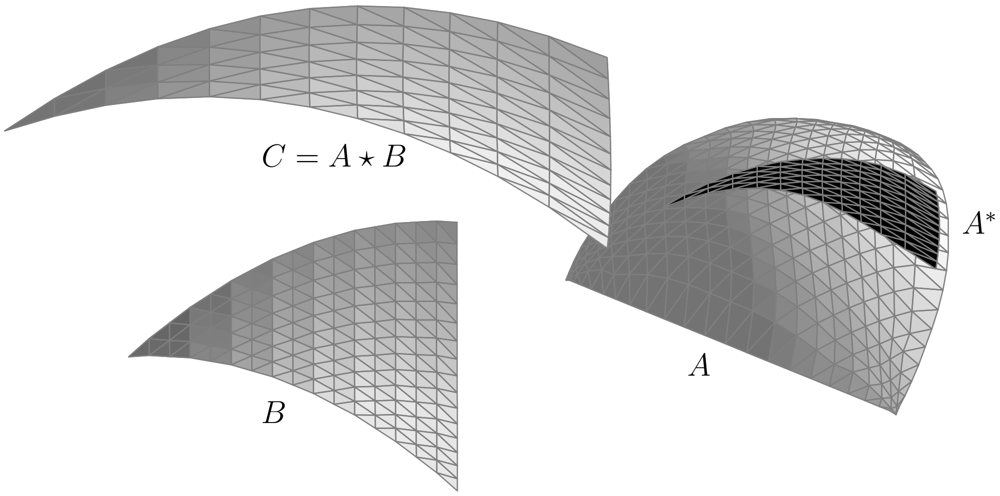


Fig. 5. Convolution surface C of a triangular patch of an LN surface of degree 6 (A) and a quadratic triangular patch B. The convolution surface is rational surface of degree 12. Only the points contained in A^* contribute to C.

- If B is a plane, the unit normal vector $\vec{\mathbf{N}}_0$ does not depend on s, t, but it is constant. Since (28) gives a single point (u, v), there is a single point $\mathbf{a}_0$ on A which corresponds to all points of B. Thus, $A \star B$ is a plane translated by the fixed vector $\mathbf{a}_0$.
- If B is a developable surface (i.e., its Gaussian curvature vanishes), ϕ maps the domain $\Omega_B \subset \mathbb{R}^2$ into a curve in the uv-plane. Thus, there is in general only a curve $\mathbf{a}(\tau) \in A$ which contributes to the construction of $A \star B$. Clearly, the convolution surface is again a developable surface.

Remark 10 Points of B with $n_3(s,t) = 0$ have no corresponding point on the LN–surface A. If there is one point with this property then, in general, there exists even a curve $c \in B$ with $n_3 = 0$ along c. The curve c is a shadow boundary of B with respect to an illumination parallel to the z-axis. In this case the convolution $A \star B$ consists of non-connected parts.

8 Conclusion

As the main result of this paper, we identified a class of free form surfaces which have rational convolution surfaces with general rational surfaces. To our knowledge, this is the first result on rational convolution surfaces of surfaces which are capable of modeling general free–form geometries.

This result may serve as the starting point for research on computing Minkowski sums of general free–form objects. While the case of two convex objects should be relatively simple, the computation of the Minkowski sum of general objects will need robust methods for detecting and trimming the inner branches of the convolution surfaces, which do not contribute to the boundary of the Minkowski sum.

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