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Functional webs for freeform architecture

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Abstract

Rationalization and construction-aware design dominate the issue of realizability of freeform architecture. The former means the decomposition of an intended shape into parts which are sufficiently simple and efficient to manufacture; the latter refers to a design procedure which already incorporates rationalization. Recent contributions to this topic have been concerned mostly with small-scale parts, for instance with planar faces of meshes. The present paper deals with another important aspect, namely long-range parts and supporting structures. It turns out that from the pure geometry viewpoint this means studying families of curves which cover surfaces in certain well-defined ways. Depending on the application one has in mind, different combinatorial arrangements of curves are required. We here restrict ourselves to so-called hexagonal webs which correspond to a triangular or tri-hex decomposition of a surface. The individual curve may have certain special properties, like being planar, being a geodesic, or being part of a circle. Each of these properties is motivated by manufacturability considerations and imposes constraints on the shape of the surface. We investigate the available degrees of freedom, show numerical methods of optimization, and demonstrate the effectivity of our approach and the variability of construction solutions derived from webs by means of actual architectural designs.

Categories and Subject Descriptors (according to ACM CCS): Computer Graphics [I.3.5]: Computational Geometry and Object Modeling—

1. Introduction

The research area of *architectural geometry* appears to have established itself as an intersection point of Geometry Processing, Optimization, Geometric Design, Architectural Design, and Engineering. This is illustrated by the diverse contributions collected by the proceedings volume [CHP*10], or the survey papers [Pot10, WP11]. Work in this area has been mostly concerned with properties of small-scale elements, like the planarity of faces. The present paper presents a computational solution of an Architectural Design problem of a different kind: Which regular patterns of long-range elements are available for the realization of freeform shapes? Such elements include structural elements and supports, floor levels, medium-range curved panels, and last, but not least, curves which are not physically realized by a single element but nevertheless are highly visible in the design.

We would like to motivate our topic with work on timber constructions pioneered by [NBM02] and continued by [PW06, WH10]. The experimental construction of Figure 2 by C. Pirazzi and Y. Weinand employs a family of *planar* curves as support elements and two families of curved beams. Both special properties of curves mentioned here – planarity and the geodesic property – facilitate fabrication. For planarity this is obvious. As to the geodesic property: Any rectangular strip of paper, and any rectangular beam

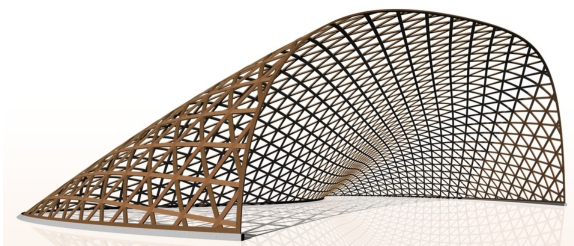


Figure 1: *A web of curved wooden panels whose shape is achieved by bending. Geometrically this is a triangle mesh with mesh polylines being geodesics on the mesh surface.*

Figure 2: *Assembling screw-laminated beams for a timber rib shell prototype based on a 2-pattern of geodesics supported by a third family of vertical elements [PW06]. This is no web. Image courtesy IBOIS (EPFL, Lausanne).*

which bends only about its weak axis, follows a geodesic curve if it is forced to lie on a surface. We particularly point to [WH10] for timber constructions which rely on this fact.

If the shape of a beam or supporting element does not enjoy any special property it is probably necessary to manufacture it by NC milling (e.g. the construction of Figure 14, left). This has also been pointed out by Pottmann et al. [PHD*10], who consider, among other patterns, webs of curves on given surfaces which are as geodesic as possible. Since true geodesic webs do not exist in general, the approximation of surfaces by geodesic webs is not always possible. The present paper follows a different approach: We optimize triangle meshes such that mesh polylines achieve certain desirable properties, but we solve an approximation problem only if there are enough degrees of freedom available.

Contributions of the present paper. We consider the following properties of space curves which are relevant for manufacturing, if that curve defines the shape of a beam or supporting element in a freeform architectural construction:

- *geodesic* curves are the shape of panels which follow a surface and whose unbent state was straight;
- *planar curves* are an easily manufacturable shape for the simple reason that most factory floors are flat;
- *circular* curves, which are part of a circle, are even more efficiently manufacturable;
- *vertical* curves (contained in planes parallel to a hypothetical z axis) are useful for support elements;
- *horizontal* curves are useful for design and for floor levels to follow.

The applications we have in mind call for three families of curves arranged in a regular pattern (see Figure 1). This arrangement is discretized as a triangle mesh of regular combinatorics, and will be called a discrete *hexagonal web* (we do not consider any other kind of web). This terminology comes from [BB38]. The three kinds of canonical mesh polylines correspond to the three family of curves. For some applications (see Figure 8) not all curves of a certain family do actually contribute to the realization of an architectural design. This does not matter: in any case we work with a regular triangle mesh and impose the desired properties either to all polylines, or only to some of them, just as the application demands. Our contributions are the following:

- we formulate web optimization as a global numerical optimization problem, with appropriate target functionals expressing the single properties enumerated above;
- in cases where the required properties leave sufficiently many degrees of freedom we show how to approximate a given surface by a web with prescribed properties;
- if the required properties are too restrictive we employ mesh optimization for form finding;
- in the case of planar webs we discuss in more detail the construction and interactive modification of webs.

Previous work on web geometry. A regular triangle mesh is a discrete version of a smooth *hexagonal web* which means three families of curves combinatorially equivalent to the families $u=$ const., $v=$ const., $u+v=$ const. of straight lines in the uv plane. For an overview we refer to the monograph [BB38] and the survey [Che82].

The discrete problem posed in the present paper is directly analogous to the smooth problem of covering a given surface by a web whose curves enjoy certain properties. This old topic of differential geometry contains many unsolved problems. For instance it is not known exactly which surfaces can be covered by a web of geodesics: [Vol29] shows that the coefficients of the 1st fundamental form must fulfill a certain 3rd order PDE, whose general solution looks hopeless. Likewise it is unknown which webs of planar curves a general surface can support (apart from the trivial cases of projecting webs of straight lines in $\mathbb{R}^2$ onto a surface). The very reduced problem of describing all webs of $\mathbb{R}^2$ formed by three linear pencils of circles has been solved only recently [Laz88, She07]. We conclude that many smooth problems of differential geometry which directly correspond to the problems studied by the present paper are unsolved, and it is hard to obtain additional insight from the smooth case.

Related work in Geometry Processing. The present paper boils down to studying architecture-relevant properties of the edges of a triangle mesh. [SS10] has similar purposes, with the *faces* as the object of interest.

There are several papers on curve networks which have the same combinatorics as those considered in the present paper. For instance, energy-minimizing networks have occurred in [WPH07]. Another example is the network of parameter lines of a global hexagonal parametrization [NPPZ10], where the optimization goal is near-ideal aspect ratios of infinitesimal triangles. Neither contribution considers functional properties in the sense we do. This leads to a completely different behaviour at combinatorial singularities, see Section 5.

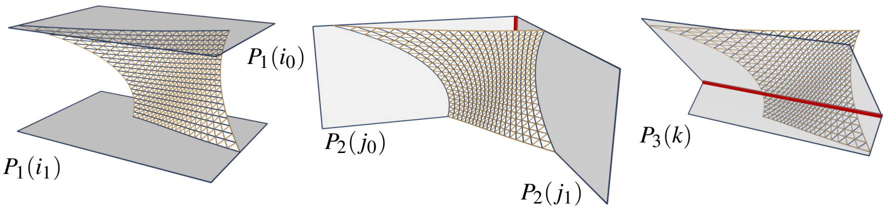


Figure 3: *The vertices $\mathbf{v}_{ij}$ of a planar web are generated by the families P_1, P_2, P_3 of planes via $\mathbf{v}_{ij} = P_1(i) \cap P_2(j) \cap P_3(c-i-j)$. Here all three families of planes are linear pencils.*

2. Planar webs

By a planar web we mean a web all of whose curves are planar. In case of regular combinatorics it is easy to give a complete parametric description of general planar webs, which is illustrated by Figure 3: Assume that the three families $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ of mesh polylines in a regular triangle mesh

$$\begin{aligned} \mathcal{L}_1 &= \{L_1(i)\}_{i\in\mathbb{Z}} && \text{(horizontal family)},\\ \mathcal{L}_2 &= \{L_2(j)\}_{j\in\mathbb{Z}} && \text{(vertical family)},\\ \mathcal{L}_3 &= \{L_3(k)\}_{k\in\mathbb{Z}} && \text{(diagonal family)}, \end{aligned}$$

are defined such that polylines $L_1(i)$, $L_2(j)$ and $L_3(k)$ intersect in a vertex "$\mathbf{v}_{ij}$" if and only if $i+j+k=c$ (for some integer c). It is no restriction to let $c=0$, but we rather have the flexibility of choosing c arbitrarily. Another way of expressing this definition is to let

$$\begin{aligned} L_1(i) &= \{\ldots, \mathbf{v}_{i0}, \mathbf{v}_{i1}, \mathbf{v}_{i2}, \ldots\},\\ L_2(j) &= \{\ldots, \mathbf{v}_{0j}, \mathbf{v}_{1j}, \mathbf{v}_{2j}, \ldots\},\\ L_3(k) &= \{\ldots, \mathbf{v}_{-k,c}, \mathbf{v}_{1-k,c-1}, \mathbf{v}_{2-k,c-2}, \ldots\}. \end{aligned} \tag{1}$$

Each polyline $L_r(i)$ is supposed to be contained in a plane $P_r(i), \ldots$, so there are three families $P_1(i)$, $P_2(j)$, $P_3(k)$ of planes where the parameters i,j,k run in the integers, and vertices arise as the intersection

$$\mathbf{v}_{ij} = P_1(i) \cap P_2(j) \cap P_3(k) \quad \text{where} \quad i+j+k=c. \tag{2}$$

So the general description of planar webs is as follows: *Select three families of planes arbitrarily and define the vertices of a triangle mesh by* (2). It is not clear how to match this parametric description to a given surface, but a simple example, where all three families are *pencils* of planes, is shown by Figure 3.

Continuous planar webs. Obviously this method can be used to describe smooth planar webs, not only discrete ones: Take three smooth families of planes $P_1(u)$, $P_2(v)$, $P_3(w)$, where u,v,w are real parameters, and define a surface by

$$x(u,v) = P_1(u) \cap P_2(v) \cap P_3(c-u-v). \tag{3}$$

Then the conditions $u=$ const. or $v=$ const. or $w=$ const. (where $u+v+w=c$) each define a planar curve on this surface. Note further that a 1-parameter family of planes generically is the family of tangent planes of a developable surface [PW01]. Thus we have shown:

Prop. 1 *A generic smooth planar web is defined by 3 developable surfaces (each being interpreted as a parametrized smooth family of its tangent planes) and vice versa. This description is unique up to a common reparametrization.*

It is no restriction, at least locally, to specify a 1-parameter family of planes as the *normal planes* of a curve: We can find such curves as orthogonal trajectories of the given planes by solving an ODE. Therefore we have:

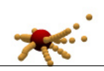
Prop. 2 *A generic smooth planar web is defined by the families of normal planes $P_1(u), P_2(v), P_3(w)$ of three parametric curves $\mathbf{c}_1(u)$, $\mathbf{c}_2(v)$ and $\mathbf{c}_3(w)$ via Equation* (3).

Trivial planar webs. It is well known that both the continuous and discrete webs contained in $\mathbb{R}^2$ whose web curves are straight lines consist of tangents of a class 3 curve. Also the converse is true: The tangents of such a curve constitute a continuous web, and an appropriate discrete sample of tangents results in a discrete web (see [BB38]; this is also discussed by [PHD*10]). By projecting such a web from an arbitrary center Z onto an arbitrary surface Φ we obtain a planar web whose vertices lie in Φ. Figure 4 illustrates how to check if a given planar web is of this 'trivial' kind.

Remark. The representation of a planar web via tangent planes of developables (Prop. 1) and normal planes of curves (Prop. 2) do not degenerate if the web is trivial: In the former case the developables are cones with vertex Z, in the latter case the curves lie in spheres with center Z.

Shape Properties. We are going to study and interactively modify webs by means of the representation given in Prop. 2. As a preparation consider a curvature-continuous curve $\mathbf{c}(t)$,

Figure 4: *A planar web is trivial if all planes spanned by web curves (set "$\mathscr{P}$" of planes) contain a common point Z. We try to find $Z = \arg\min \sum_{P\in\mathscr{P}} dist(\mathbf{x},P)^2$ (red) and visualize orthogonal projection of Z onto all planes (yellow). If Z exists this point cloud is concentrated in Z. Here it is not $\Longrightarrow$ no trivial web.*



which is given in Bézier or B-spline form as follows:

$$\mathbf{c}(t) = \sum_j B_j(t)\mathbf{p}_j \quad (t \in [\alpha,\beta],\ j \in \{a,\dots,b\}). \quad (4)$$

We require that the derivatives of curves are expressible in terms of *differences* of control points such that

$$\frac{d\mathbf{c}}{dt} = \sum_j \widetilde{B}_j(t)\sigma_j \Delta\mathbf{p}_j, \quad \Delta\mathbf{p}_j = \mathbf{p}_{j+1} - \mathbf{p}_j, \quad (5)$$

$$\widetilde{B}_j(t) \geq 0, \quad \sum_j \widetilde{B}_j(t) = \text{const.} = 1, \quad \sigma_j > 0. \quad (6)$$

This is true for several classes of curves:

- $\mathbf{c}$ is a Bézier curve: Here B_j, $\widetilde{B}_j$ are Bernstein polynomials of degrees n and $n-1$, resp. We take $[\alpha,\beta] = [0,1]$, indices range in $j \in \{0,\dots,n\}$, and $\sigma_j = n$.
- $\mathbf{c}$ is a *B-spline curve* of degree $n \geq 3$ over a knot vector $\{u_i\}$: Here $\widetilde{B}_j$ are B-spline basis functions of degree $n-1$ over the same knot vector, and $\sigma_j = \frac{n}{u_{j+n+1} - u_{j+1}}$.

The construction of a planar web surface $x(u,v)$ from three families of planes involves computing the intersection point of planes: There is usually an entire curve of parameter values (u,v) where the intersection point is undefined (i.e., lies at infinity). However, it is possible to use the shape properties of freeform curves expressed by (6) to derive a condition which ensures that this does not happen, and the web surface is a proper surface of Euclidean space:

Prop. 3 *The continuous 'planar' web surface*

$$x(u,v) = P_1(u) \cap P_2(v) \cap P_3(c-u-v)$$

defined by the three families of normal planes $P_1(u)$, $P_2(v)$, $P_3(w)$ of curves $\mathbf{c}_1(u)$, $\mathbf{c}_2(v)$, $\mathbf{c}_3(w)$ does not have points at infinity if and only if the derivatives of curves obey

$$\det(\dot{\mathbf{c}}_1(u), \dot{\mathbf{c}}_2(v), \dot{\mathbf{c}}_3(c-u-v)) \neq 0. \quad (7)$$

If $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ are Bézier or B-spline curves, with control points

$$\{\mathbf{p}_j^{(1)}\}_{j=a_1,\dots,b_1},\ \{\mathbf{p}_j^{(2)}\}_{j=a_2,\dots,b_2},\ \{\mathbf{p}_j^{(3)}\}_{j=a_3,\dots,b_3},$$

such that Equations (4), (5), (6) hold (for appropriate choices of intervals $[\alpha_i,\beta_i]$ and index ranges $[a_i,b_i]$) , then (7) is fulfilled in the domain $u \in [\alpha_1,\beta_1]$, $v \in [\alpha_2,\beta_2]$, $c-u-v \in [\alpha_3,\beta_3]$, if for all j,k,l,

$$\det(\Delta\mathbf{p}_j^{(1)}, \Delta\mathbf{p}_k^{(2)}, \Delta\mathbf{p}_l^{(3)}) > 0. \quad (8)$$

The same conclusion holds if all determinants are negative.

Proof The expression for the intersection point $x(u,v)$ of planes is rational. Its denominator is the determinant of normal vectors which occurs in (7). We avoid points at infinity if (7) is nonzero. In the Bézier/spline case, (7) expands to

$$\sum_{j,k,l} f_{jkl}(u,v,c-u-v)\sigma_j^{(1)}\sigma_k^{(2)}\sigma_l^{(3)} \det(\Delta\mathbf{p}_j^{(1)}\Delta\mathbf{p}_k^{(2)}\Delta\mathbf{p}_l^{(3)}),$$

where $f_{jkl}(u,v,w) = \widetilde{B}_j^{(1)}(u)\widetilde{B}_k^{(2)}(v)\widetilde{B}_l^{(3)}(w)$. (6) implies that the functions f_{jkl} are nonnegative and their sum equals 1, so the determinant under consideration is a convex combination of positive values, if (8) holds. □

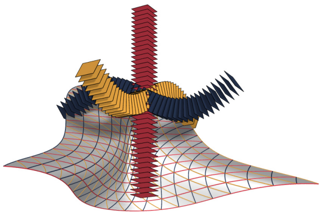


Figure 5: *Continuous planar web generated by 3 families of planes orthogonal to a curve (the correspondence between web curves and planes is defined by colors).*

Cor. 4 *The conclusion of Prop. 3 remains true if condition (8) is required only for such indices j,k,l were the plane $u+v+w=c$ intersects the support box of $f_{jkl}(u,v,w)$. A further reduction of the sufficient conditions occurs if the parametric domain of the web surface in the u,v-plane is reduced.*

Proof Only such triples (j,k,l) contribute to (7). □

3. Global optimization of discrete webs

This section deals in detail with the different conditions imposed on webs, and how to formulate them in a way which is useful in optimization. Section 4 below deals with the local and interactive modification of planar webs, which is also formulated as an optimization problem.

The combinatorial setup is rather simple: We consider triangle meshes (V,E,F) whose vertices have valence 6 except for the boundary. We have already discussed their indexing and how the edges form the three types of mesh polylines $L_1(i)$, $L_2(j)$, $L_3(k)$ — see Equ. (1). Accordingly we have three kinds of edges:

$$E = E_1 \cup E_2 \cup E_3,$$

where the edges contained in set E_i are used to form the mesh polylines "L" contained in family $\mathcal{L}_i$. We use the notation $L \in \mathcal{L}_i$ and $\mathbf{v} \in L$. Any submesh of this mesh inherits the three families of mesh polylines.

For optimization of a web, we use the vertex coordinates as variables, together with additional variables depending on the geometric properties we want to achieve (see the respective sections below). Basically we minimize a target functional of the form

$$f = \sum \lambda_{\text{property},j} f_{\text{property}_j}(\mathcal{L}_j) + \sum \lambda_{\text{fair},j} f_{\text{fair}_j}(\mathcal{L}_j) + \lambda_{prox} f_{prox} + \lambda_{bdry} f_{bdry}. \quad (9)$$

Here $f_{\text{property}_j}(\mathcal{L}_j)$ refers to a functional which penalizes deviation of family $\mathcal{L}_j$ from a certain 'property'. Similarly $f_{\text{fair}_j}(\mathcal{L}_j)$ means a fairness/regularization functional which applies to family $\mathcal{L}_j$. The symbols f_{prox}, f_{bdry} penalize deviation of the web from the target shape and its boundary, respectively. The λ's represent weights.

Remark. In our examples often not all polylines of family $\mathcal{L}_i$ are required to possess a certain geometric property, only some of them are. In this case the target functional (9) remains formally unchanged, but we consider a smaller set $\mathcal{L}_i$.

Fairing. For regularization we employ *differences* of consecutive vertices of polylines: $f_{fair}(\mathscr{L})$ is one of

$$f_{fair,II}(\mathscr{L}) = \sum_{L\in\mathscr{L}} \Big(\sum_{\substack{\text{consecutive vertices}\\ \mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2 \text{ in } L}} \|\mathbf{v}_0 - 2\mathbf{v}_1 + \mathbf{v}_2\|^2 \Big),$$

$$f_{fair,III}(\mathscr{L}) = \sum_{L\in\mathscr{L}} \Big(\sum_{\substack{\text{cons. vertices}\\ \mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3 \text{ in } L}} \|\mathbf{v}_0 - 3\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3\|^2 \Big).$$

Each summand corresponds to a sequence of two or three consecutive edges in a polyline. It turns out that 2^{nd} order differences penalize uneven spacing of vertices to a higher degree than 3^{rd} order differences, but otherwise minimizing 3^{rd} order differences is to be preferred since it does not try to make polylines straight. For more details see Figure 11.

Reference Shape. In order to penalize the deviation of a web surface "Ψ" from a reference shape Φ we should use $\int_{\mathbf{x}\in\Phi} \operatorname{dist}(\mathbf{x},\Psi)^2$, for f_{prox}, which is not readily computable. In the manner of the well known ICP algorithm we replace $\operatorname{dist}(\mathbf{x},\Psi)^2$ by $\|\mathbf{x}-\mathbf{x}'\|^2$, where $\mathbf{x}'$ is the closest-point projection of $\mathbf{x}$ onto the web surface (in each iteration step of our optimization procedure, the points $\mathbf{x}'$ are updated). Using the squared tangent plane distance would be a better approximation of $\operatorname{dist}(\mathbf{x},\Psi)^2$ if $\mathbf{x}$ is close to Ψ, but it would lead to shrinking during optimization. For computational purposes the integral is computed via a dense sample of Φ. This term is used for Figures 1, 7, 8, 10, 14. An alternative is to exchange the roles of web surface and reference surface, and use $\sum_{\mathbf{v}\in V} \operatorname{dist}(\mathbf{v},\Phi)^2$ for f_{prox} instead. Here $\mathbf{v}'$ is orthogonal projection of $\mathbf{v}$ onto Φ.

Reference Boundary. If the web surface Ψ is to be close to a curve $\partial\Phi$ we derive f_{bdry} from $\int_{\mathbf{x}\in\partial\Phi} \operatorname{dist}(\mathbf{x},\Psi)^2$, with $\operatorname{dist}(\mathbf{x},\Psi)$ replaced by distance to the closest-point projection, similar to f_{prox}. An alternative functional (forcing the boundary of the web to be close to $\partial\Phi$) is $f_{bdry} = \sum \operatorname{dist}(\mathbf{v}, T_{\mathbf{v}'})^2$ where $\mathbf{v}'$ means orthogonal projection onto the boundary curve, and $T_{\mathbf{v}'}$ is a tangent to that curve. It is used for Figure 9.

3.1. The planar property

This section describes how to optimize a web such that one or more families of polylines consist of planar curves. If each polyline L of one selected family $\mathscr{L}$ is to be contained in a plane, we consider the equation $\mathbf{x}^T\mathbf{n}_L - u_L = 0$ of that plane and introduce the unit normal vector $\mathbf{n}_L$ and the coefficient u_L as auxiliary variables. The geometry functional corresponding to family $\mathscr{L}$ then reads

$$f_{planar}(\mathscr{L}) = \textstyle\sum_{L\in\mathscr{L}} \sum_{\mathbf{v}\in L} (\mathbf{v}^T\mathbf{n}_L - u_L)^2,$$

where $\|\mathbf{n}_L\|^2 = 1$. The auxiliary variables are initialized by finding a best approximating plane for each polyline L (via principal component analysis).

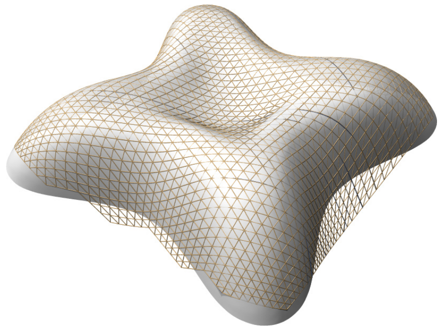


Figure 6: *Approximating a surface with a planar web. Its quality is illustrated by the fact that when fitting exactly planar curves (light) to the mesh polylines (dark) the deviation is very small in all places and consequently the dark curves are hardly ever visible (design surface taken from the top of Lilium Tower, Warszaw, by Zaha Hadid Architects).*

Counting degrees of freedom. In order to gain an overview on the available degrees of freedom, we count the number of conditions imposed on a web if one family $\mathscr{L}$ is required to be planar. We have 3 additional degrees of freedom for each auxiliary plane and one condition of containment of a vertex in a plane. So the effective number of conditions imposed is

$$C_{planar}(\mathscr{L}) = \#\{\text{Vertices of } \mathscr{L}\} - 3\#\{\text{Planes of } \mathscr{L}\},$$
$$\text{provided } \#L > 3 \text{ for all } L\in\mathscr{L}.$$

I.e., we count only polylines which have at least 4 vertices. Imposing planarity on all three families $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$ yields the degrees of freedom for the vertices:

$$\text{d.o.f.} = 3\#V - \textstyle\sum C_{planar}(\mathscr{L}_j) \geq 3\sum \#\{\text{Planes of } \mathscr{L}_j\}.$$

We conclude that in general we have sufficiently many remaining degrees of freedom to be able to satisfy additional requirements such as proximity of a web to a reference shape. We can therefore expect to be able to solve an approximation problem even beyond the 'trivial' planar webs which exist on all surfaces (for an example, see Figure 6). Of course this counting of degrees of freedom takes neither fairness nor suitability for practical purposes into account.

An alternative local functional. Co-planarity of the vertices contained in a polyline L implies co-planarity of all possible choices of 4 successive vertices: With $\delta(\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$ for the distance of straight lines $\mathbf{v}_0 \vee \mathbf{v}_1$ and $\mathbf{v}_2 \vee \mathbf{v}_3$ we may define an alternative target functional by letting

$$\widetilde{f}_{planar}(\mathscr{L}) = \sum_{L\in\mathscr{L}} \Big(\sum_{\substack{\text{consecutive vertices}\\ \mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3 \text{ in } L}} \delta(\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)^2 \Big).$$

The function $\widetilde{f}_{planar}$ leads to faster computation than f_{planar}, but $\widetilde{f}_{planar} = 0$ does not imply $f_{planar} = 0$ if 4 or more vertices in a row lie on a straight line. Even if $f_{fair} \to \min$ tends to evenly distribute deviation from straight lines and so this is

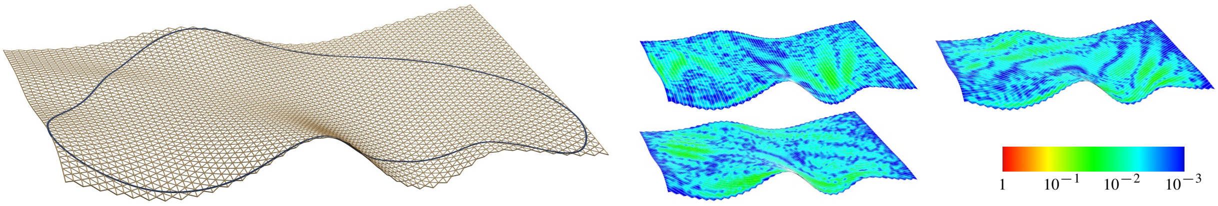


Figure 7: *A geodesic web through a given boundary curve (Ex. 5). Left: Optimization result before trimming. Other figures: Color coded values of the geodesic curvature of each family of web polylines (bounding box size 1).*

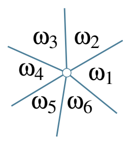
unlikely, we still should test for planarity after optimization with $\widetilde{f}_{planar}$. We did not use this functional in our examples.

Special positions of planes. For applications it may be desirable that the polylines in one family are parallel to a fixed plane which is considered 'horizontal'. In this case the target functional is much simplified, since the normal vector $\mathbf{n}_L$ of such planes is a constant. Similarly we can require that planes are parallel to a fixed direction which is considered 'vertical'. Assuming the z axis is vertical, this yields the simplification that normal vectors have the form $\mathbf{n}_L = (n_1, n_2, 0)^T$. Several examples in this paper make use of such special positions of planes.

3.2. The geodesic property

When dealing with *geodesic* mesh polylines we employ the concept of *straightest geodesics* introduced by [PS98]. Consider a valence 6 vertex and number the six edges $e_1, \ldots, e_6$ emanating from this vertex cyclically such that

$$e_i, e_{i+3} \in E_i \quad (i = 1, 2, 3).$$



With the angles $\omega_i = \sphericalangle(e_i, e_{i+1})$ (indices modulo 6) the condition that the edges e_i, e_{i+3} are part of a *geodesic* polyline then reads

$$\omega_i + \omega_{i+1} + \omega_{i+2} = \omega_{i-1} + \omega_{i-2} + \omega_{i-3}. \qquad (10)$$

Accordingly we consider the target functional

$$f_{geod}(\mathscr{L}_i) = \frac{1}{e^2} \sum\nolimits_{\mathbf{v} \in V} \Big(\sum\nolimits_{j=1,2,3} \omega_{i+j}(\mathbf{v}) - \omega_{i-1-j}(\mathbf{v}) \Big)^2,$$

where e is an average edgelength, the index $i \in \{1,2,3\}$ denotes the family, and the dependence of angles from the vertex $\mathbf{v}$ is indicated by the notation $\omega_i(\mathbf{v})$. If we require all three families to be geodesic, then it is not difficult to see that

$$(10) \text{ holds for all } i \iff \omega_i = \omega_{i+3} \text{ for all } i.$$

This simplified condition leads to the functional

$$f^{all}_{geod} = \frac{1}{e^2} \sum\nolimits_{\mathbf{v} \in V} \sum\nolimits_{i=1,2,3} \big(\omega_i(\mathbf{v}) - \omega_{i+3}(\mathbf{v})\big)^2.$$

Note that both $f_{geod}(\mathscr{L}_i)$, f^{all}_{geod} are local functionals.

Counting degrees of freedom. If the family $\mathscr{L}$ is required to be geodesic, we impose one scalar condition on each *interior* vertex – for boundary vertices we cannot evaluate (10). The effective number of conditions therefore equals

$$C_{geod}(\mathscr{L}) = \#\{\text{interior vertices of } \mathscr{L}\}.$$

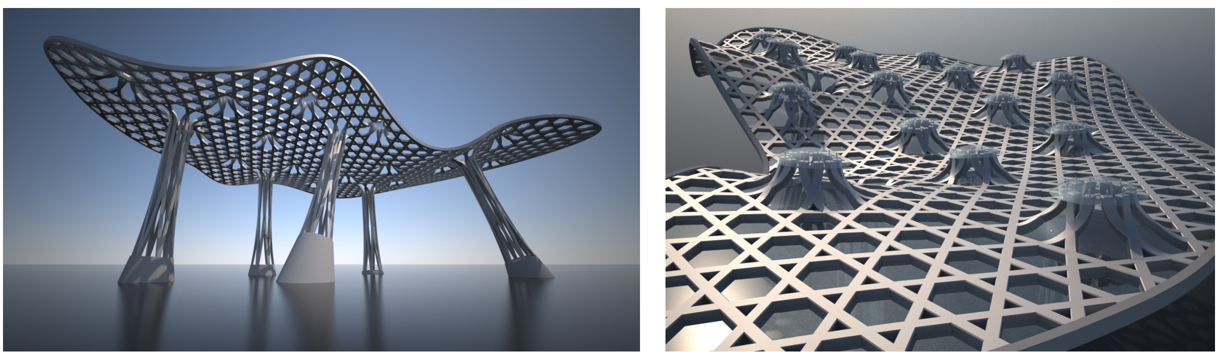


Figure 8: *By deleting some polylines in the the geodesic web of Fig. 7 we obtain a tri-hex structure which serves as the basis of an architectural design. The greater part of beams in this example is to be made from layers, each individually bent into shape (which is possible because of the geodesic property). This is a significant advance over NC milling necessary otherwise.*

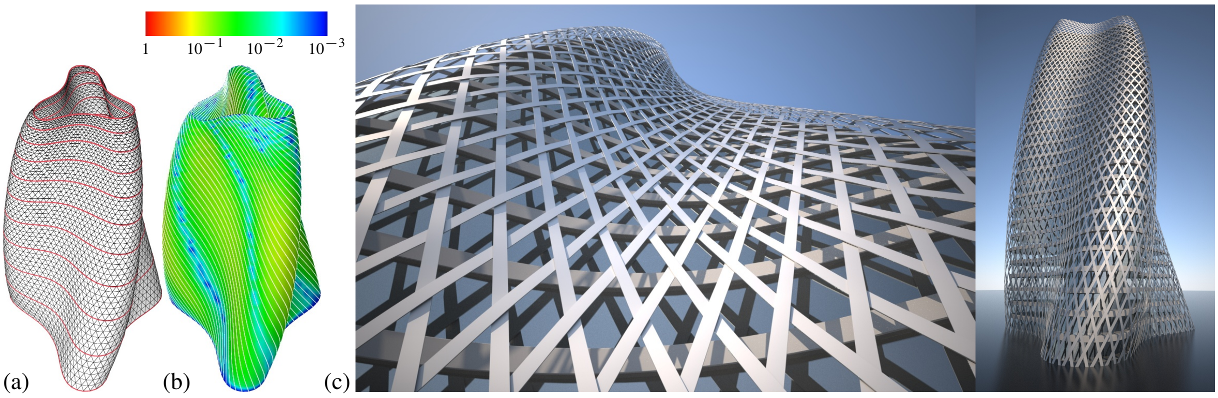


Figure 9: *A web where families $\mathcal{L}_1, \mathcal{L}_2$ are geodesic, and every sixth polyline of the third family of polylines is lying in a horizontal plane. (a) shows fitted 'horizontal' curves in red. (b) Geodesic curvatures of $\mathcal{L}_1$ are small (color coded values). The same is true for $\mathcal{L}_2$ (not shown). (c) Architectural design based on this web. The diagonal families of curves are, for all practical purposes, geodesics, so we introduce additional high frequency bending of small amplitude to obtain a 'weaving' design.*

Requiring all three families $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ to be geodesic consequently leads to a d.o.f. count of

$$\text{d.o.f.} = 3\#V - \sum C_{geod}(\mathcal{L}_i) = 3\#\{\text{bdry vertices}\},$$

i.e., as many degrees of freedom as there are coordinates of boundary vertices. We can therefore expect that *a geodesic web is uniquely determined by its boundary*. For practical purposes, however, this uniqueness translates to insufficient degrees of freedom, and one solves the boundary value problem in a different way (see Example 5 and Figure 7).

Example 5 In order to find a geodesic web through a given boundary curve $\partial\Phi$ we initialize optimization by a triangle mesh which lies in a plane which best-approximates $\partial\Phi$ (found by principal component analysis) and subsequently employ a linear combination of functionals f_{geod}, f_{fair}, and f_{bdry} . A result is shown by Figure 7. The architectural design of Figure 8 is based on this web: Deleting selected polylines yields a tri-hex structure which is manufacturable by pure bending, if made from thin beams.

Example 6 We illustrate modified mesh combinatorics and the combination of different properties by means of a mesh with cylinder topology, where $\mathcal{L}_1$ consists of m closed polylines of n vertices each. We impose the geodesic condition on $\mathcal{L}_2$ and $\mathcal{L}_3$, which means $2n(m-2)$ conditions. Further we retain in $\mathcal{L}_1$ only every k-th polyline and require it to lie in a horizontal plane. The number of imposed conditions is $n-1$ per 'horizontal' polyline, which adds up to $(n-1)\lfloor \frac{m}{k} \rfloor$. Thus,

$$\text{d.o.f.} = n(m+4) - (n-1)\left\lfloor \frac{m}{k} \right\rfloor.$$

In the extremal case $k = 1$ this equals $4n + m$, which is still positive. Figure 9 shows an example where $k = 6$, and where the geometric shape has been achieved by means of proximity to a reference surface. This web is basis of the architectural 'weaving' design of Figure 9, right. We should remark that requiring the same z coordinate for polylines of the 3rd family led to an even distribution of 'floor levels', it was not necessary to prescribe those z coordinates.

3.3. The Circular Property.

This section discusses how to optimize a web such that the polylines of one family or possibly of more families are circular arcs. Conceptually this is very similar to the planar property discussed in Section 2, by introducing the circles' defining parameters as auxiliary variables. We want to treat straight lines as special cases of circles which are in no way degenerate, so we cannot use the radius as a variable. Instead we employ a certain normalization of the circle's equation proposed by [Pra87]: Using coordinates ξ_1, ξ_2 w.r.t. some

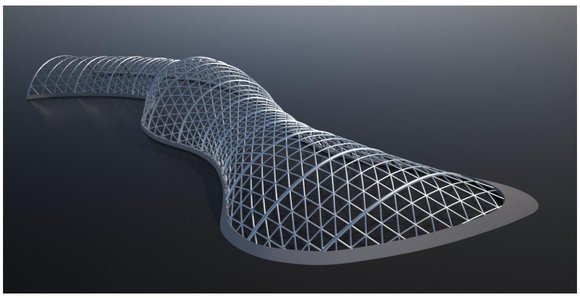


Figure 10: *A web with 2 geodesic families, and one family where every 4th polyline lies in a vertical plane and is circular. The circular property imposes $2(n-3)$ conditions on n points, so it is usually not possible to require it for all polylines of a family, if also other constraints are to be fulfilled.*

Fig	$\#V$	T [sec]	max κ_g	w	m_{planar}	w	m_{circular}	w	$m_{\text{fair,II}}$	w	$m_{\text{fair,III}}$	w	m_{bdry}	w	m_{prox}	w
1	7563	100	0.06	$6{\cdot}10^{-6}$					0.002	0.1			$1{\cdot}10^{-4}$	1	$2{\cdot}10^{-4}$	1
4, 6	5155	66			0.009	9					0.01	0.01			0.002	1
7, 8	10074	214	0.05	$8{\cdot}10^{-5}$					0.001	1			0.001	1		
9	13187	176	0.2	$8{\cdot}10^{-5}$	0.01	2.3			0.005	9	0.003	1	0.004	2.3	0.04	1
10	12537	210	0.4	$2{\cdot}10^{-5}$	0.003	2.3	0.002	2.3	0.002	16	0.002	1	0.006	0.4	0.005	0.4
14	13396	1002			0.01	9					0.01	1	0.009	1	0.009	1

Figure 11: *Quality of optimization. We show the maximum geodesic curvature of polylines which are optimized to be geodesic; as well as the maximum values m_{planar}, m_{circular}, etc., of expressions whose squares contribute to f_{planar}, f_{circular}, etc. If both $f_{\text{fair,II}}$ and $f_{\text{fair,III}}$ are used, then the former applies to geodesic polylines. The weights given to these functionals in the optimization are derived from the value "w" associated to each column: w is the weight of the respective term occurring in (9), multiplied with the number of summands which contribute to that term.*

orthonormal coordinate frame $\mathbf{e}_1, \mathbf{e}_2$, it reads

$$f_{\mathbf{b}}(\xi_1,\xi_2) = b_0(\xi_1^2+\xi_2^2) + b_1\xi_1 + b_2\xi_2 + b_3 = 0$$
$$\text{where } \mathbf{b} = (b_0,\dots,b_3), \quad b_1^2+b_2^2-4b_0b_3 = 1. \qquad (11)$$

We have $\|\nabla f_{\mathbf{b}}\| = 1$ whenever $f_{\mathbf{b}} = 0$, so the value $|f_{\mathbf{b}}(\xi_1,\xi_2)|$ is a good approximation of the distance of the point (ξ_1,ξ_2) from the circle $[f_{\mathbf{b}} = 0]$.

The location in space of such a circle is determined by a Cartesian coordinate frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ attached to the origin of $\mathbb{R}^3$ and by the information that the circle lies in the plane $\xi_3 = u$. Supposing the polylines of family $\mathscr{L}$ shall be circular, we introduce the collection

$$\{(\mathbf{b}_L, \mathbf{e}_{1,L}, \mathbf{e}_{2,L}, \mathbf{e}_{3,L}, u_L)\}_{L\in\mathscr{L}}$$

of auxiliary variables, which determine a circle for each polyline $L \in \mathscr{L}$. These variables come with the constraints that $\mathbf{b}_L$ satisfies (11), and that $(\mathbf{e}_{1,L}, \mathbf{e}_{2,L}, \mathbf{e}_{3,L})$ is an orthonormal frame. In order to diminish the number of side conditions we encode these frames as unit quaternions.

The contribution $f_{\text{circular}}(\mathscr{L})$ to the target functional which penalizes deviation from the circular property accordingly is defined by

$$f_{\text{circular}}(\mathscr{L}) = \sum_{L\in\mathscr{L}} \sum_{\mathbf{v}\in L} f_{\mathbf{b}_L}(\mathbf{e}_{1,L}^T\mathbf{v}, \mathbf{e}_{2,L}^T\mathbf{v})^2 + (\mathbf{e}_{3,L}^T\mathbf{v} - u_L)^2.$$

The auxiliary variables, which specify a circle associated with a polyline L, are initialized by first fitting a plane to L, projecting L onto it, and then using the metric of (11) to fit a circle to the vertices obtained in this way [Che10].

Remark. The frame associated with a circle is not unique. In fact there is no continuous assignment $\{\text{circles}\} \longmapsto \{\text{frames}\}$ because it would give rise to a continuous nonzero vector field on the unit sphere (map unit axis vector of circles of fixed radius to first basis vector of frame). This is forbidden by the hairy ball theorem. However, this non-uniqueness does not hinder optimization.

Example 7 Figure 10 shows a 'rationalization' example. A reference surface is approximated by a web where the families $\mathscr{L}_1, \mathscr{L}_2$ are geodesic, and every 4th polyline of the third family is circular, lying in a vertical plane.

Remark. We might require a polyline L to be congruent to part of a master curve C. This is easily incorporated into our optimization via a rotation R and a translation vector $\mathbf{t}$ such that for all vertices $\mathbf{v} \in L$, we have $\operatorname{dist}(R\mathbf{v}+\mathbf{t}, C) \approx 0$. However we cannot in general expect success: Imposing this condition on a polyline of n vertices is equivalent to $2n$ conditions on the $3n+6$ unknowns R, $\mathbf{t}$ and the vertices of L.

3.4. Implementation

Our academic implementation loads a reference geometry and an initial triangle mesh. The user specifies which family is to enjoy which geometric property (and also if that property applies only to every k-th polyline). There is no universal rule for the choice of weights in (9); the user is expected to set the weights and adjust them according to the quality of the result which has been achieved.

We employ a Gauss-Newton method with Levenberg-Marquardt regularization [MNT04]. All required 1st derivatives are computed analytically. The linear system solved in each round of optimization is sparse, since each summand of the target functional (9) involves only a few variables. We therefore make use of CHOLMOD for sparse Cholesky factorization [CDHR08]. Figure 11 shows optimization quality and details on the choice of weights.

4. Interactive Deformation of Planar Webs

We use the representation of continuous planar webs according to Prop. 2 for their interactive modification. There are two issues to consider: Firstly, the condition of planarity globally couples all three families of web polylines and so a general planar web has no localized deformations. Secondly, generating a surface $x(u,v)$ by intersection of planes is rather unstable numerically and is prone to zero denominators. We must therefore obey the conditions given by Prop. 3.

Optimization Setup. We represent an initial planar web surface $x^*(u,v)$ as stated by Prop. 3 and use the control points $\mathbf{p}_j^{(1)}$, $\mathbf{p}_k^{(2)}$, $\mathbf{p}_l^{(3)}$ of the three defining curves as the variables in optimization.

Fig.	ctrl.pts	#var	$M+N$	fairing samples	T [sec]
13 b1,c1	15	45	7×7	$7^2\times7^2=2401$	6
13 b2,c2	21	63	6×6	$6^2\times8^2=2304$	3

Figure 12: *Data on the number of variables and samples used in the optimization of a web deformation, as well as computing time on a laptop with a 2.4GHz CPU.*

For certain parameter values (u_i,v_i) $(i=1,\dots,M)$ the desired location $\mathbf{t}_i$ of the surface point $x(u_i,v_i)$ is given. We imagine the user has selected $x^*(u_i,v_i)$ and dragged it to its new location $\mathbf{t}_i$. To accommodate the condition of *proximity* of the modified web to its initial state, we uniformly cover the web with N additional points $x^*(u_i,v_i)$ $(i=M+1,\dots,M+N)$ and set a desired location $\mathbf{t}_i=x^*(u_i,v_i)$. Optimization uses the target functional

$$f=\lambda_{\mathrm{fair}}f_{\mathrm{fair}}+\sum\nolimits_{i=1}^{M+N}w_i\|x(u_i,v_i)-\mathbf{t}_i\|^2, \qquad (12)$$

which is minimized under the side-conditions of Prop. 3, or rather Cor. 4. Here weights $w_1,\dots,w_M$ of selected points are set to 1, while the weights $w_{M+1},\dots,w_{M+N}$ of the remaining targets are smaller — the closer the distance of such a point to the selection set, the smaller the weight. In this way the web will be optimized such that points which are distant from the user's selection remain where they are.

The symbol f_{fair} in (12) means a regularization term: We apply the previously defined 2nd order and 3rd order fairing terms $f_{fair,II}$ and $f_{fair,III}$ to a discrete web generated by a fine regular sample of the parameter domain.

Figure 13 shows results obtained for different choices of weights. In all cases we have $w_i=\omega(\delta_i)$, where 'parametric' weighting means $\delta_i=\min_{j=1}^M\|(u_i,v_i)-(u_j,v_j)\|$, and 'Euclidean' weighting means $\delta_i=\min_{j=1}^M\|x(u_i,v_i)-x(u_j,v_j)\|$. With user-defined constants α,β $(0\le\beta\ll\alpha\le1)$, we used $\omega(\delta)=(\alpha\delta_{red}+\beta(1-\delta_{red}))^2$, with $\delta_{red}=\frac{\delta-\delta_{\min}}{\delta_{\max}-\delta_{\min}}$.

Implementation. For minimizing (12) under the side conditions (8) we employ sequential quadratic programming. We use the *NLopt* library [Joh11], based on [Kra94]. Since the number of unknowns is small, computation time depends largely on the number $M+N$ of target destinations and the number of samples employed in fairing (see Figure 12).

5. Discussion

There are limitations to web optimization, especially to finding webs such that both a reference surface is approximated and geometric properties are fulfilled. This is due to the global and coupled nature of constraints. For example, Figure 14 shows a surface which is difficult to approximate by a planar web in its entirety.

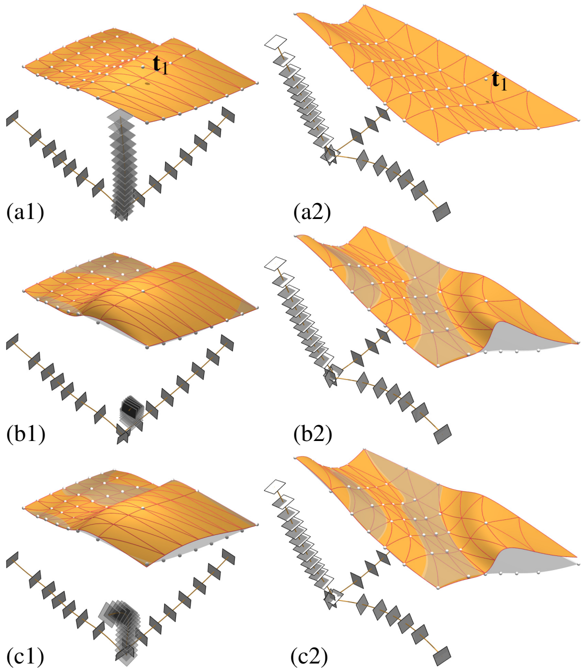


Figure 13: *Deformation of continuous planar webs defined by the normal planes of curves (shown by Figs. (a1), (a2)). One target point $\mathbf{t}_1$ is shown off the surface, and N auxiliary target points can be seen resting in it. The resulting deformations are shown in figures (b1), (b2), using 'Euclidean' weights, and (c1), (c2) for 'parametric' weights.*

Local Control. For the same reason, local control can be expected to be available only to a limited extent. For the case of nontrivial planar webs this is demonstrated by Figure 13. For geodesic webs covering a given surface Φ_0, a dimension count in the smooth case shows that we cannot expect that locally deformed surfaces Φ_t $(0\le t<\varepsilon)$ are still covered by a geodesic web (By [Vol29], Φ_t must satisfy a certain 4th order PDE. If the deformation is local, all Φ_t's share values and derivatives along a common boundary. This usually implies that all Φ_t's are the same).

Singularities. A combinatorial limitation is the regularity of meshes we use. It is of course possible to apply local functionals such as f_{geod}^{all}, $\widetilde{f}_{planar}$, f_{prox}, and f_{bdry} to more general meshes, but this does not make sense in most cases. For instance, requiring the geodesic property near a vertex of valence $\neq 6$ leads to a singularity of the surface geometry. This implication "combinatorial singularity $\Longrightarrow$ geometric singularity" follows from the existence of geodesic mesh polylines which run side by side before encountering the singularity and which diverge immediately afterwards, implying a concentration of Gaussian curvature (cf. [PHD*10]). Other work on regular triangular and hexagonal parametrizations such as [NPPZ10] does not suffer from this problem.

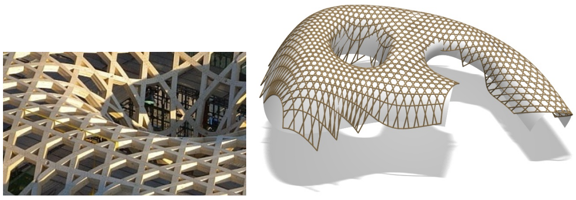


Figure 14: *Web optimization tasks may be hard to solve. Motivated by the roof structure of the Yeoju Golf Club by Shigeru Ban (left) we cover the design surface at right by a planar web. The result is not useful for aesthetic reasons.*

Conclusion and Future Research. We have shown how to approach the optimization of webs algorithmically and we have demonstrated some of their applications in construction-aware design of freeform architecture. It is important that the optimization of webs can be used as a design tool. We have in particular studied the planar webs in this respect, but in fact every web optimization, where the approximation constraint cannot be met, is an instance of *form finding*. Another one, where we do not go into details, is shown by Figure 15.

As to future research, we repeat that the mathematical theory of webs is in some places rather incomplete. In particular sufficient conditions on the existence of webs of certain types which cover given surfaces are largely missing. We believe that this subject area is important for applications and there will be demand for optimization solutions which are more specific and which combine webs with other geometric aspects of architectural geometry.

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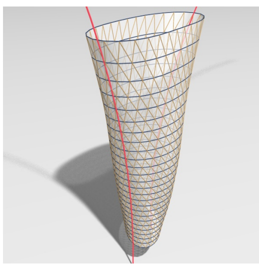

Figure 15: *Evolution of a web with cylinder topology from a base curve and two guiding curves. Here we have 1 family of planar 'horizontal' curves, and 2 families of geodesics.*