

# Conchoid surfaces of spheres

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## Abstract

The conchoid of a surface  $F$  with respect to given fixed point  $O$  is roughly speaking the surface obtained by increasing the radius function with respect to  $O$  by a constant. This paper studies *conchoid surfaces of spheres* and shows that these surfaces admit rational parameterizations. Explicit parameterizations of these surfaces are constructed using the relations to pencils of quadrics in  $\mathbb{R}^3$  and  $\mathbb{R}^4$ . Moreover we point to remarkable geometric properties of these surfaces and their construction.

*Keywords:* sphere, pencil of quadrics, rational conchoid surface, rational polar representation, Viviani's curve.

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## 1. Introduction

The conchoid is a classical geometric construction and dates back to the ancient Greeks. Given a planar curve  $C$ , a fixed point  $O$  (focus point) and a constant distance  $d$ , the conchoid  $D$  of  $C$  with respect to  $O$  at distance  $d$  is the (Zariski closure of the) set of points  $Q$  in the line  $OP$  at distance  $d$  of a moving point  $P$  varying in the curve  $C$ ,

$$D = \{Q \in OP \text{ with } P \in C \text{ and } \overline{QP} = d\}^*, \quad (1)$$

where the asterisk denotes the Zariski closure. For a more formal definition of conchoids in terms of diagrams of incidence we refer to [10, 11].

The definition of the conchoid surface to a given surface  $F$  in space with respect to a given point  $O$  and distance  $d$  follows analogous lines.

We aim at studying real rational surfaces in 3-space whose conchoid surfaces are also rational and real. A surface  $F \subset \mathbb{R}^3$  will be represented by a polar representation  $\mathbf{f}(u, v) = \rho(u, v)\mathbf{k}(u, v)$ , where  $\mathbf{k}(u, v)$  is a parameterization of the unit sphere  $S^2$ . Without loss of generality we assume  $O = (0, 0, 0)$ . Consequently their conchoid surfaces  $F_d$  for varying distance  $d$  admit the polar representation  $\mathbf{f}_d(u, v) = (\rho(u, v) \pm d)\mathbf{k}(u, v)$ .

Since we want to determine classes of surfaces whose conchoid surfaces for varying distances are rational, we focus at rational polar surface representations. Then the 'base' surface  $F$  and its conchoids  $F_d$  correspond to the same rational parameterization  $\mathbf{k}(u, v)$  of the unit sphere  $S^2$ . The following definition excludes possibly occurring cases where  $F$  and  $F_d$  are rational, but their rational parameterizations  $\mathbf{f}$  and/or  $\mathbf{f}_d$  are not corresponding to a rational representation  $\mathbf{k}(u, v)$  of  $S^2$ .

**Definition 1.** *A surface  $F$  is called rational conchoid surface with respect to the focus point  $O = (0, 0, 0)$  if  $F$  admits a rational polar representation  $\rho(u, v)\mathbf{k}(u, v)$ , with a rational radius function  $\rho(u, v)$  denoting the distance function from  $O$  to  $F$  and a rational parameterization  $\mathbf{k}(u, v)$  of  $S^2$ .*

### 1.1. Contribution

The main contribution of this article is the study of the conchoid surfaces of spheres. We prove that a sphere  $F$  in  $\mathbb{R}^3$  admits a rational polar representation  $\mathbf{f}(u, v) = \rho(u, v)\mathbf{k}(u, v)$  with a rational

radius function  $\rho(u, v)$  and a particular rational parameterization  $\mathbf{k}(u, v)$  of the unit sphere  $S^2$ , independently of the relative position of the sphere  $F$  and the focus point  $O$ . This implies that the conchoids  $G$  of  $F$  with respect to any focus in  $\mathbb{R}^3$  admit rational parameterizations.

It is remarkable that an analogous result to this contribution for spheres does not exist for circles and conics in  $\mathbb{R}^2$ . The conchoid curves of conics  $C$  are only rational if either  $O \in C$  or  $O$  coincides with one of  $C$ 's focal points. In some ways this property is comparable to conics and quadrics. As known the offset curves of ellipses and hyperbolas are non-rational, whereas the offset surfaces of all regular quadrics admit real rational parameterizations, see for instance [3, 6].

Two constructions to prove the main result are presented. The first one uses the cone model being introduced in Section 1.2 and studies a pencil of quadrics in  $\mathbb{R}^4$ . This construction is explicit and leads to a surprisingly simple solution and a rational polar representation of a sphere. The second approach investigates pencils of quadrics in  $\mathbb{R}^3$  containing a sphere and a cone of revolution whose base locus is a rational quartic with rational distance from  $O$ .

### 1.2. The cone model

Let  $F$  be a surface in  $\mathbb{R}^3$  and let  $G$  be its conchoid surface at distance  $d$  with respect to the origin  $O = (0, 0, 0)$  as focal point. The construction of the conchoid surfaces  $G$  of the 'base' surface  $F$  is performed as follows. Consider Euclidean 4-space  $\mathbb{R}^4$  with coordinate axis  $x, y, z$  and  $w$ , where  $\mathbb{R}^3$  is embedded in  $\mathbb{R}^4$  as the hyperplane  $w = 0$ . Consider the quadratic cone  $D : x^2 + y^2 + z^2 - w^2 = 0$  in  $\mathbb{R}^4$ . Further, let  $A$  be the cylinder through  $F$ , whose generating lines are parallel to  $w$ . Note that  $A$  as well as  $D$  are three-dimensional manifolds in  $\mathbb{R}^4$ . The conchoid construction of the 'base' surface  $F$  is based on the study of the intersection  $\Phi = A \cap D$ , which is typically a two-dimensional surface in  $\mathbb{R}^4$ .

For a given parameterization  $\mathbf{f}(u, v)$  of  $F$  in  $\mathbb{R}^3$ , the cylinder  $A$  through  $F$  admits the representation  $\mathbf{a}(u, v, s) = (f_1, f_2, f_3, 0) + s(0, 0, 0, 1)$ . Let  $F$  be a rational surface and  $\mathbf{f}(u, v)$  be rational. If the intersection  $\Phi = A \cap D$  is a rational surface in  $\mathbb{R}^4$ , then it is obvious that  $F$  admits a rational polar representation. Let  $\varphi(a, b) = (\varphi_1, \dots, \varphi_4)(a, b)$  be a rational representation of  $\Phi$  in  $\mathbb{R}^4$ , then  $(\varphi_1, \varphi_2, \varphi_3)(a, b)$  is obviously a rational polar representation of  $F$ . Since  $\varphi_4^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2$  holds,  $\mathbf{k} = 1/\varphi_4(\varphi_1, \varphi_2, \varphi_3)$  is a rational parameterization of  $S^2$  and  $\rho(a, b) = \varphi_4(a, b)$  is a rational radius function of  $F$ . We summarize the construction.

**Theorem 2.** *The rational conchoid surfaces  $F \subset \mathbb{R}^3$  are in bijective correspondence to the rational 2-surfaces in the quadratic cone  $D : x^2 + y^2 + z^2 - w^2 = 0$  in  $\mathbb{R}^4$ .*

*Proof:* We proved already that for a rational surface  $\Phi \subset D$ , its orthogonal projection  $(\varphi_1, \varphi_2, \varphi_3)$  onto  $\mathbb{R}^3$  is a rational conchoid surface with rational radius function  $\varphi_4$ . Conversely, any rational conchoid surface  $F$  with respect to  $O$  is defined by a rational polar parameterization  $\rho(u, v)\mathbf{k}(u, v)$ , with  $\mathbf{k} = (k_1, k_2, k_3) \in S^2$ . The corresponding surface  $\Phi \subset D$  is represented by  $\varphi(u, v) = \rho(k_1, k_2, k_3, 1)(u, v)$ .  $\square$

The quadratic cone  $D$  possesses universal parameterizations and we may use them to specify all possible rational parameterizations of rational conchoid surfaces. The construction starts with rational universal parameterizations of the unit sphere  $S^2$ . Following [2] we choose four arbitrary polynomials  $a(u, v)$ ,  $b(u, v)$ ,  $c(u, v)$  and  $d(u, v)$  without common factor. Let

$$\alpha = 2(ac + bd), \beta = 2(bc - ad), \gamma = a^2 + b^2 - c^2 - d^2, \delta = a^2 + b^2 + c^2 + d^2,$$

then  $\mathbf{k}(u, v) = \frac{1}{\delta}(\alpha, \beta, \gamma)$  is a rational parameterization of the unit sphere  $S^2$ . Thus  $\varphi(u, v) = \rho(u, v)(\alpha, \beta, \gamma, \delta)$  with a non-zero rational function  $\rho(u, v)$  is a rational parameterization of a two-dimensional surface  $\Phi \subset D$ . Consequently,

$$\mathbf{f}(u, v) = \rho(u, v) \left( \frac{\alpha}{\delta}, \frac{\beta}{\delta}, \frac{\gamma}{\delta} \right) (u, v) = \rho(u, v)\mathbf{k}(u, v)$$

is a rational polar representation of a rational conchoid surface  $F$  in  $\mathbb{R}^3$ , with  $\rho(u, v)$  as radius function and  $\mathbf{k}(u, v)$  as rational parameterization of the unit sphere  $S^2$ . It is sufficient to consider polynomials and the construction reads as follows.

**Corollary 3.** *Given six relatively prime polynomials  $a(u, v)$ ,  $b(u, v)$ ,  $c(u, v)$ ,  $d(u, v)$ , and  $r(u, v)$  and  $s(u, v)$ , a universal parameterization of a rational 2-surface  $\Phi \subset D$  in  $\mathbb{R}^4$  is given by*

$$\varphi(u, v) = \frac{r}{s} (2(ac + bd), 2(bc - ad), a^2 + b^2 - c^2 - d^2, a^2 + b^2 + c^2 + d^2) (u, v). \quad (2)$$

Consequently, a universal rational parameterization of a rational conchoid surface reads

$$\mathbf{f}(u, v) = \frac{r}{s(a^2 + b^2 + c^2 + d^2)} (2(ac + bd), 2(bc - ad), a^2 + b^2 - c^2 - d^2) (u, v). \quad (3)$$

This is a general result about all rational parameterizations of rational conchoid surfaces. For a particular given rational surface  $F$  it is difficult to decide whether the intersection  $\Phi = D \cap W$  admits rational parameterizations or not. Typically the surface  $\Phi$  is not rational. Nevertheless, there are interesting non-trivial cases where  $\Phi$  admits rational parameterizations.

In [5] it has been proved that conchoids of rational ruled surfaces  $F$  are rational. We sketch the proof using the cone model  $D$  and Theorem 2. If  $F$  is a ruled surface, the cylinder  $A \subset \mathbb{R}^4$  carries a one-parameter family of planes parallel to the  $w$ -axis. These planes pass through the generating lines of  $F$ . This implies that typically the intersection  $\Phi = A \cap D$  carries a one-parameter family of conics obtained as intersections of the mentioned planes with  $D$ . This family of conics is rational, and it is known ([4, 7, 8]) that there exist rational parameterizations  $\varphi(u, v)$  of  $\Phi$ . Thus the conchoids of real rational ruled surfaces are rational.

In this context we mention a trivial but useful statement which we prove for completeness.

**Lemma 4.** *Given a rational curve  $C$  with parameterization  $\mathbf{c}(t)$  on a rotational cone  $D$ , then the distance  $\|\mathbf{c}(t) - \mathbf{v}\|$  between the curve  $C$  and the vertex  $\mathbf{v}$  of  $D$  is a rational function.*

*Proof:* We use a special coordinate system with  $\mathbf{v}$  at the origin, and  $z$  as rotational axis of  $D$ . This implies that  $D$  is the zero set of  $x^2 + y^2 - \gamma^2 z^2$ . Without loss of generality we let  $\gamma = 1$ . The given curve  $C$  admits therefore a rational parameterization  $\mathbf{c}(t) = (c_1, c_2, c_3)(t)$  satisfying  $c_1^2 + c_2^2 = c_3^2$ . Obviously one obtains  $\|\mathbf{c}(t)\| = \sqrt{2}c_3(t)$  being rational.  $\square$

## 2. Conchoids of spheres

Given a sphere  $F$  in  $\mathbb{R}^3$  and an arbitrary focus point  $O$ , the question arises if there exists a rational representation  $\mathbf{f}(u, v)$  of  $F$  with the property that  $\|\mathbf{f}(u, v)\|$  is a rational function of the parameters  $u$  and  $v$ . To give a constructive answer to this question we describe an approach using the cone-model presented in Section 1.2. Later on in Section 3 we study a different method working in  $\mathbb{R}^3$  directly. There are several relations between these methods which will be discussed along their derivation.

Let  $F$  be the sphere with center  $\mathbf{m} = (m, 0, 0)$  and radius  $r$ , and let  $O = (0, 0, 0)$ . Thus  $F$  is given by

$$F : (x - m)^2 + y^2 + z^2 - r^2 = 0. \quad (4)$$

### 2.1. Special Cases

If  $m = 0$ , the center of  $F$  coincides with  $O$ . In this trivial situation the conchoid surface of  $F$  is reducible and consists of two spheres. If additionally  $d = r$ , one of them degenerates to  $F$ 's center.

If  $m^2 - r^2 = 0$ , the focal point  $O$  is contained in  $F$ . To construct a rational polar representation, we make the ansatz  $\mathbf{f}(u, v) = \rho(u, v)\mathbf{k}(u, v)$  with  $\mathbf{k}(u, v) = (k_1, k_2, k_3)(u, v)$  and  $\|\mathbf{k}(u, v)\| = 1$  and an unknown radius function  $\rho(u, v)$ . Plugging this into (4), we obtain a rational polar representation with rational radius function  $\rho(u, v) = 2mk_1(u, v)$ . Note that in this case the conchoid is irreducible and rational.

## 2.2. Pencil of quadrics in $\mathbb{R}^4$

Consider the Euclidean space  $\mathbb{R}^4$  with coordinate axes  $x, y, z$  and  $w$  and let  $\mathbb{R}^3$  be embedded as the hyperplane  $w = 0$ . Let a sphere  $F \subset \mathbb{R}^3$  be defined by (4) and  $O = (0, 0, 0)$ . To study the general case we assume  $m \neq 0$  and  $m^2 \neq r^2$ . The equation of the cylinder  $A \subset \mathbb{R}^4$  through  $F$  with  $w$ -parallel lines agrees with the equation of  $F$  in  $\mathbb{R}^3$ ,

$$A : (x - m)^2 + y^2 + z^2 - r^2 = 0. \quad (5)$$

Consider the pencil  $Q(t) = A + tD$  of quadrics in  $\mathbb{R}^4$ , spanned by  $A$  and the quadratic cone  $D : x^2 + y^2 + z^2 = w^2$  from Section 1.2. Any point  $\bar{X} = (x, y, z, w) \in D$  has the property that the distance from  $X = (x, y, z)$  to  $O$  in  $\mathbb{R}^3$  equals  $w$ . We study the geometric properties of the del Pezzo surface  $\Phi = A \cap D$  of degree four, the base locus of the pencil of quadrics  $Q(t)$ . According to Theorem 2, the sphere  $F$  is a rational conchoid surface exactly if  $\Phi$  admits rational parameterizations.

Besides  $A$  and  $D$  there exist two further singular quadrics in  $Q(t)$ . These quadrics are obtained for the zeros  $t_1 = -1$  and  $t_2 = r^2/\gamma^2$  of the characteristic polynomial

$$\det(A + tD) = -(1 + t)^2 t (\gamma^2 t - r^2), \text{ with } \gamma^2 = m^2 - r^2 \neq 0. \quad (6)$$

The quadric corresponding to the twofold zero  $t_1 = -1$  is a cylinder

$$R : w^2 - 2mx + m^2 - r^2 = 0. \quad (7)$$

Its directrix is a parabola in the  $xw$ -plane and its two-dimensional generators are parallel to the  $yz$ -plane. The singular quadric  $S$  corresponding to  $t_2 = r^2/\gamma^2$  is a quadratic cone and reads

$$S : \left( x - \frac{m^2 - r^2}{m} \right)^2 + y^2 + z^2 = \frac{r^2}{m^2} w^2.$$

Its vertex is the point  $O' = (\frac{m^2 - r^2}{m}, 0, 0, 0)$ . The intersections of  $S$  with three-spaces  $w = c$  are spheres  $\sigma(c)$ , whose top view projections in  $w = 0$  are centered at  $O'$  and their radii are  $rc/m$ . The intersections of  $D$  with three-spaces  $w = c$  are spheres  $d(c)$  whose top view projections in  $w = 0$  are centered at  $O$  with radii  $c$ . The intersections  $k(c) = \sigma(c) \cap d(c)$  of these spheres ( $w = c$ ) are circles in planes  $x = (c^2 + m^2 - r^2)/(2m)$ . Thus  $\Phi$  contains a family of conics, whose top view projections are the circles  $k(c)$ . The conics in  $\Phi$  are contained in the planes

$$\varepsilon(c) : x = \frac{c^2 + m^2 - r^2}{2m}, w = c.$$

The half opening angle  $\delta$  of  $D$  with respect to the  $w$ -axis is  $\pi/4$ , thus  $\tan \delta = 1$ . The half opening angle  $\sigma$  of  $S$  is given by  $\tan \sigma = r/m$ , see Figure 1(a). Applying the scaling

$$(x', y', z', w') = (fx, fy, fz, w), \text{ with } f = \frac{r}{m}$$

in  $\mathbb{R}^4$  maps  $D$  to a congruent copy of  $S$ . Consider a point  $\bar{X} = (x, y, z, w)$  in  $\Phi = A \cap D$  and its projection  $X = (x, y, z)$  in  $F$ . The distance  $\text{dist}(X, O)$  of  $X$  to  $O$  in  $\mathbb{R}^3$  is  $w$ . For the distance  $\text{dist}(X, O')$  between  $X$  and  $O'$  we consequently obtain

$$\text{dist}(X, O') = \frac{r}{m} \text{dist}(X, O), \text{ for all } X \in F. \quad (8)$$

*Remark on the circle of Apollonius.* Note that  $O'$  is the inverse point of  $O$  with respect to the sphere  $F$ . It is an old result by Apollonius Pergaeus (262–190 b.c.) that the set of points  $X$  in the plane having constant ratio of distances  $f = d/d'$ , with  $d = \text{dist}(O, X)$  and  $d' = \text{dist}(O', X)$ , from two given fixed points  $O$  and  $O'$ , respectively, is a circle  $k$ , see Fig. 1(b). Rotating  $k$  around the line  $OO'$  gives the sphere  $F$  and  $O$  and  $O'$  are inverse points with respect to  $F$  (and the circle  $k$ ).

If we consider a varying constant ratio  $f$ , one obtains a family of spheres  $F(f)$  with inverse points  $O$  and  $O'$  which form an elliptic pencil of spheres. Their centers are on the line  $OO'$ . The ratio 1 ( $d = d'$ ) corresponds to the bisector plane of  $O$  and  $O'$ .

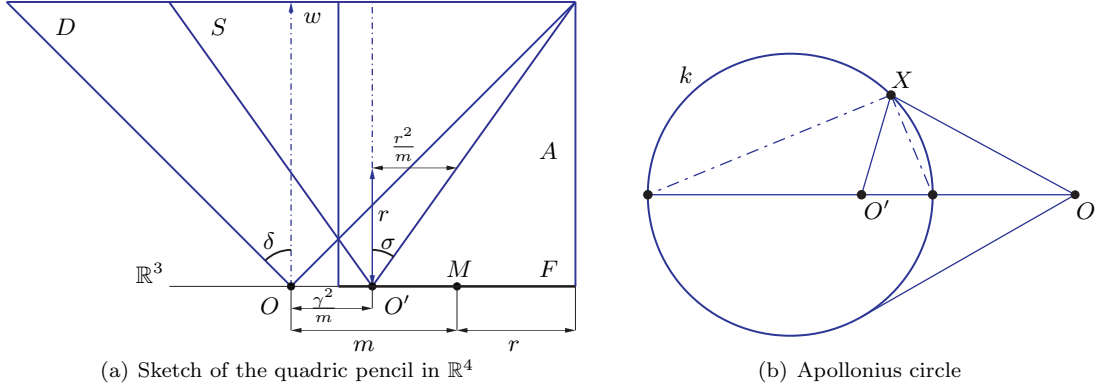


Figure 1: Pencil of quadrics in  $\mathbb{R}^4$  and Apollonius circle

### 2.3. A rational quartic on the sphere

The pencil of quadrics  $Q(t)$  in  $\mathbb{R}^4$  spanned by the cylinder  $A$  and the cone  $D$  contains the cylinder  $R$ . Expressing the variable  $x$  from (7) one gets

$$x = \frac{w^2 + m^2 - r^2}{2m}, \quad (9)$$

and inserting this into  $D$  results in the polynomial

$$\alpha(w) : 4m^2(y^2 + z^2) + p(w) = 0, \text{ with } p(w) = w^4 - 2w^2(m^2 + r^2) + (m^2 - r^2)^2. \quad (10)$$

Considering  $y$  and  $z$  as variables,  $\alpha(w)$  is a one-parameter family of conics (circles) in the  $yz$ -plane, depending *rationally* on the parameter  $w$ . The circles  $\alpha(w)$  do not possess real points for all  $w$ , but there exist intervals determining families of real circles  $\alpha(w)$ . To obtain real circles one has to perform a re-parameterization  $w(u)$  within an appropriate interval. The factorization of  $p(w)$  reads

$$p(w) = (w + a)(w - a)(w + b)(w - b), \text{ with } a = m + r, \text{ and } b = m - r.$$

If  $O$  is outside of  $F$ , thus  $m > r$ , the polynomial  $-p(w)$  is positive in the interval  $[m - r, m + r]$ . Thus a possible re-parameterization is

$$w(u) = \frac{au^2 + b}{1 + u^2} = \frac{u^2(m + r) + m - r}{1 + u^2}. \quad (11)$$

Otherwise we could re-parameterize over another appropriate interval. Additionally we note that if  $O$  is inside of  $F$ , the inverse point  $O'$  is outside of  $F$ . Since equation (8) holds for the distances of a point  $X \in F$  to  $O$  and  $O'$ , we can exchange roles and perform the computation for the point  $O'$ .

We return to the family of conics  $\alpha(w)$ . Substituting (11) into (10) leads to a family of real conics

$$\alpha(u) : y^2 + z^2 = \frac{4r^2u^2}{m^2(1 + u^2)^4}(au^2 + m)(mu^2 + b). \quad (12)$$

We are looking for rational functions  $y(u)$  and  $z(u)$  satisfying (12) identically. Therefore we introduce auxiliary variables  $\tilde{y}$  and  $\tilde{z}$  by the relations  $y = 2\tilde{y}ru/(m(1 + u^2)^2)$  and  $z = 2\tilde{z}ru/(m(1 + u^2)^2)$ . We obtain  $\tilde{y}^2 + \tilde{z}^2 = (au^2 + m)(mu^2 + b)$ . Factorizing the left and right hand side of this equation results in a linear system to determine  $\tilde{y}$  and  $\tilde{z}$ ,

$$\begin{aligned} \tilde{y} + i\tilde{z} &= (\sqrt{au} + i\sqrt{m})(\sqrt{mu} + i\sqrt{b}), \\ \tilde{y} - i\tilde{z} &= (\sqrt{au} - i\sqrt{m})(\sqrt{mu} - i\sqrt{b}). \end{aligned}$$

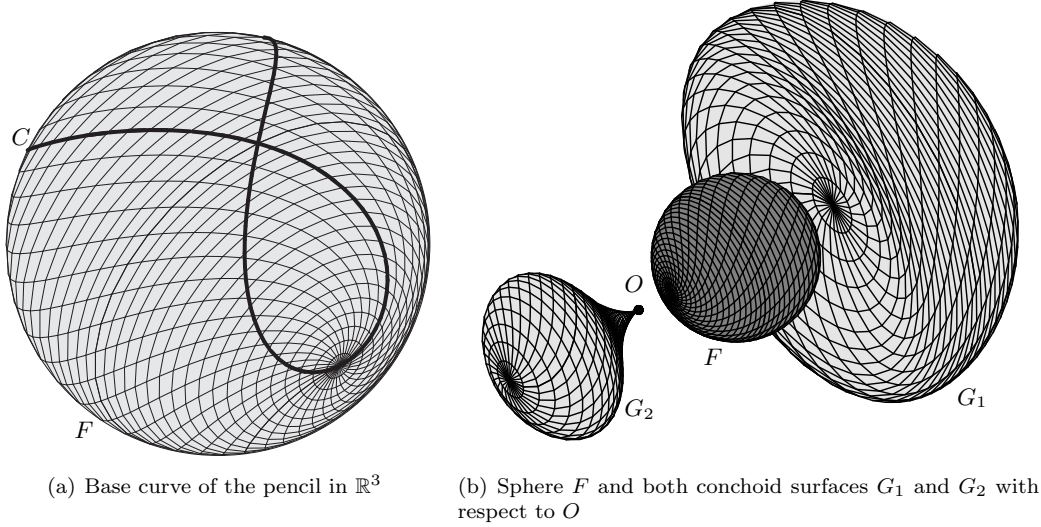


Figure 2: Rational polar representation of a sphere and its conchoid surfaces

The solution  $\tilde{y} = \sqrt{m}(\sqrt{au^2 - \sqrt{b}})$ ,  $\tilde{z} = u(m + \sqrt{ab})$  finally leads to

$$y(u) = \frac{2r\sqrt{mu}}{m(1+u^2)^2} (\sqrt{au^2 - \sqrt{b}}), \text{ and } z(u) = \frac{2ru^2}{m(1+u^2)^2} (m + \sqrt{ab}), \quad (13)$$

which is a rational parameterization of a curve in the  $yz$ -plane, following the family of conics  $\alpha(w)$ .

We note that any real rational family of conics possesses real rational parameterizations, see for instance [8, 9]. The solution (13) together with (9) determines a curve  $C \subset F$  which possesses the non-constant rational distance function

$$\|\mathbf{c}(u)\| = w(u) = \frac{u^2(m+r) + (m-r)}{1+u^2} \quad (14)$$

with respect to  $O$ . Its parameterization is

$$\mathbf{c}(u) = \frac{1}{m(1+u^2)^2} \begin{pmatrix} u^4m(m+r) + 2u^2(m^2 - r^2) + m(m-r) \\ 2r\sqrt{mu}(u^2\sqrt{m+r} - \sqrt{m-r}) \\ 2ru^2(m + \sqrt{m^2 - r^2}) \end{pmatrix}. \quad (15)$$

**Theorem 5.** *Let  $F$  be a sphere and let  $O$  be an arbitrary point in  $\mathbb{R}^3$ . Then there exists a rational quartic curve  $C \subset F$  and a rational parameterization  $\mathbf{c}(u)$  of  $C$  such that the non-constant distance of  $C$  to  $O$  is a rational function in the curve parameter  $u$ .*

Rotating  $C$  around the  $x$ -axis leads to a rational polar representation  $\rho(u, v)\mathbf{k}(u, v)$  of  $F$  with rational distance function  $\rho(u, v) = w(u)$  from  $O$ . The quartic curve  $C$  together with this parameterization is illustrated in Fig. 2(a). We summarize the presented construction.

**Theorem 6.** *Spheres in  $\mathbb{R}^3$  admit rational polar representations with respect to any focus point  $O$ . This implies that the conchoid surfaces of spheres admit rational parameterizations. The construction is based on rational quartic curves on  $F$  with rational distance from  $O$ .*

*Remark on the rationality.* The construction performed in Section 2.3 yields a rational polar representation  $\rho(u, v)\mathbf{k}(u, v)$  of the sphere  $F$ . Typically a point  $X \in F$  corresponds to two points in the parameter domain.

The conchoid surface  $G$  of  $F$  at distance  $d$  typically consists of two surfaces  $G_1$  and  $G_2$  (see Fig. 2(b)), which admit the rational parameterizations  $\mathbf{g}_{1,2} = (\rho(u, v) \pm d)\mathbf{k}(u, v)$ . The conchoid  $G = G_1 \cup G_2$  is an irreducible algebraic surface of order six. It is *not bi-rationally* equivalent to the projective plane but each component admits *improper* rational parameterizations. These components  $G_1$  and  $G_2$  are called *uni-rational*.

We consider the sphere  $F$  with center  $\mathbf{m} = (3/2, 0, 0)$  and radius  $r = 1$ . The conchoid surface  $G = G_1 \cup G_2$  for variable distance  $d$  is given by the in general irreducible equation

$$\begin{aligned} G: & (x^2 + y^2 + z^2)(4(x^2 + y^2 + z^2) - 12x + 5)^2 \\ & + d^2(40(x^2 + y^2 + z^2) - 144x^2 + 96x(x^2 + y^2 + z^2) - 32(x^2 + y^2 + z^2)^2) \\ & + 16d^4(x^2 + y^2 + z^2) = 0. \end{aligned} \quad (16)$$

*Remark on the rational quartic.* The rational quartic  $C$  on  $F$  is of course not unique but depends on the re-parameterization (11). An admissible quadratic re-parameterization determines a rational quartic  $C$  on  $F$ , with rational norm  $\|\mathbf{c}\| = w(u)$ . Rotating  $C$  around the line  $OM$  yields a rational polar representation of at least some part of  $F$ .

*Remark on the conchoid surfaces of quadrics.* Since the construction of the conchoid surfaces of spheres is based on rational parameterizations of a del Pezzo surface  $\Phi$  as base locus of a quadric pencil in  $\mathbb{R}^4$ , an analogous construction works for the conchoid surfaces of quadrics  $F$  in  $\mathbb{R}^3$ . Applying particular rational transformations fixing  $O$  we can achieve that  $F$  is given by the normal form  $F : \pm 1 \pm a^2x^2 + b^2y^2 + c^2z^2 = 0$ . Rational parameterizations of  $\Phi$  are constructed either via a rational family of conics in  $\Phi$  or by the method proposed in [1]. These constructions and geometric investigations will be elaborated in a separate contribution.

#### 2.4. Pencil of quadrics in $\mathbb{R}^3$

The quartic curve  $C$  from (15) on the sphere  $F$  is the base locus of a pencil of quadrics  $F + \lambda K$  in  $\mathbb{R}^3$ , spanned by  $F$  and the projection cone  $K$  of  $C$  from its double point  $\mathbf{s}$ , see Fig. 3. The double point  $\mathbf{s}$  is located in the symmetry plane of  $C$  and in the polar plane of the origin  $O$  with respect to  $F$ . Its coordinates are

$$\mathbf{s} = \frac{1}{m}(\gamma^2, 0, r\gamma) \text{ with } \gamma^2 = m^2 - r^2. \quad (17)$$

The pencil  $F + \lambda K$  contains two further singular quadrics which are obtained for the zeros  $\lambda_1 = 1/m$  and  $\lambda_2 = -1/\gamma$  of the characteristic polynomial

$$\det(F + \lambda K) = r^2(m\lambda - 1)(\gamma\lambda + 1).$$

Corresponding to  $\lambda_1$  there is a parabolic cylinder  $P$  with  $y$ -parallel generating lines passing through  $C$ . Corresponding to  $\lambda_2$  we find the rotational cone  $L$  through  $C$  with vertex  $O$ .

To give explicit representations for the quadrics we use homogeneous coordinates  $\mathbf{y} = (1, x, y, z)^T$ . Since there should not be any confusion, we use same notations for the quadric  $X$  and its coordinate matrix  $X$  appearing in the homogeneous quadratic equation  $\mathbf{y}^T \cdot X \cdot \mathbf{y} = 0$ . The coefficient matrices  $F$  and  $K$  read

$$F = \begin{pmatrix} m^2 - r^2 & -m & 0 & 0 \\ -m & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} \gamma^3 & -\gamma m & 0 & 0 \\ -\gamma m & \gamma & 0 & r \\ 0 & 0 & -m & 0 \\ 0 & r & 0 & -\gamma \end{pmatrix}. \quad (18)$$

An elementary computation shows that  $K$  is a cone of revolution with opening angle  $\pi/2$  and  $\tilde{\mathbf{a}} = (m + \gamma, 0, r)$  denotes a direction vector of its axis. The cone  $L$  through  $C$  with vertex at  $O$  is

again a cone of revolution, whose axis is parallel to  $\tilde{\mathbf{a}}$ . The axis of the cross section parabola of  $P$  in the  $xz$ -plane is orthogonal to  $\tilde{\mathbf{a}}$ , see Fig. 3(a). The coefficient matrices  $L$  and  $P$  are

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r \\ 0 & 0 & m + \gamma & 0 \\ 0 & -r & 0 & 2\gamma \end{pmatrix}, P = \begin{pmatrix} \gamma^2(m + \gamma) & -m(m + \gamma) & 0 & 0 \\ -m(m + \gamma) & m + \gamma & 0 & r \\ 0 & 0 & 0 & 0 \\ 0 & r & 0 & m - \gamma \end{pmatrix}. \quad (19)$$

A trigonometric parameterization of the quartic  $C$  is obtained by intersecting the cone  $K$  with one quadric of the pencil  $F + \lambda K$ , for instance  $F$ . Let  $\mathbf{a}$  be a unit vector in direction of  $K$ 's axis, and  $\mathbf{b}$  and  $\mathbf{c}$  complete it to an orthonormal basis in  $\mathbb{R}^3$ . A trigonometric parameterization of  $K$  is given by

$$\mathbf{k}(t, \lambda) := \mathbf{s} + \lambda(\mathbf{a} + (\mathbf{g} \cos t + \mathbf{h} \sin t)), \text{ with} \\ \mathbf{a} = \frac{1}{\sqrt{2m(m+\gamma)}}(m + \gamma, 0, r), \mathbf{g} = (0, -1, 0), \text{ and } \mathbf{h} = \frac{1}{\sqrt{2m(m+\gamma)}}(r, 0, -(m + \gamma)).$$

Thus  $K$  admits the explicit parameterization

$$\mathbf{k}(t, \lambda) = \frac{1}{2m\sqrt{m(m+\gamma)}} \begin{pmatrix} 2\gamma^2\sqrt{m(m+\gamma)} + \lambda\sqrt{2m(m+\gamma+r\sin t)} \\ -2\lambda m\sqrt{m(m+\gamma)} \cos t \\ 2r\gamma\sqrt{m(m+\gamma)} + \lambda\sqrt{2m(r-(m+\gamma)\sin t)} \end{pmatrix}.$$

Finally, a trigonometric parameterization of the quartic  $C$  follows by

$$\mathbf{c}(t) = \frac{1}{2m} \begin{pmatrix} (m + r \sin t)^2 + \gamma^2 \\ \sqrt{2}\sqrt{m(m+\gamma)} \cos t (\gamma - m - r \sin t) \\ r(m + \gamma) \cos^2 t \end{pmatrix}, \text{ with } \|\mathbf{c}(t)\| = m + r \sin t. \quad (20)$$

The correspondence of the trigonometric parameterization and its norm with the expressions (15) and (14) in terms of rational functions is realized by the substitutions  $\sin t = (u^2 - 1)/(u^2 + 1)$  and  $\cos t = 2u/(u^2 + 1)$  and some rearrangement of the equations. Section 2.5 discusses relations to Viviani's curve (or Viviani's window). This particular quartic has a similar shape and its pencil of quadrics has similar properties. Viviani's curve has an additional symmetry.

*Remark.* The inversion with center  $O$  at the sphere which intersects the given sphere  $F$  perpendicularly, maps the sphere  $F$  onto itself. Analogously this inversion fixes the rotational cone  $L$ . Thus the quartic intersection curve  $C = F \cap L$  remains fixed as a whole, but of course not point-wise. The product of the distances  $\text{dist}(O, P)$  and  $\text{dist}(O, P')$  of two inverse points  $P \in F$  and  $P' \in F$  equals  $\sqrt{m^2 - r^2}$ . This property follows from the elementary tangent-secant-theorem of a circle.

### 2.5. Relations to Viviani's curve

The quartic curve  $C$ , the base locus of the pencil of quadrics  $F + tK$ , can be considered as generalization of Viviani's curve  $V$ . This particularly well known curve  $V$  is the base locus of a pencil of quadrics, spanned by a sphere  $F$  and a cylinder of revolution  $L$  touching  $F$  and passing through the center of  $F$ . The pencil of quadrics of Viviani's curve also contains a right circular cone  $K$  with vertex in  $V$ 's double point and opening angle  $\pi/2$ , and further a parabolic cylinder  $P$ . Viviani's curve  $V$  is obtained from  $C$  by a limit. To perform this in detail, choose the inverse point  $O' = (\frac{m^2 - r^2}{m}, 0, 0)$  as origin. The parameterization (20) of  $C$  becomes

$$\mathbf{c}(t) = \frac{1}{2m} \begin{pmatrix} r^2(1 + \sin^2 t) + 2mr \sin t \\ \sqrt{2}\sqrt{m(m+\gamma)} \cos t (\gamma - m - r \sin t) \\ r(m + \gamma) \cos^2 t \end{pmatrix}. \quad (21)$$



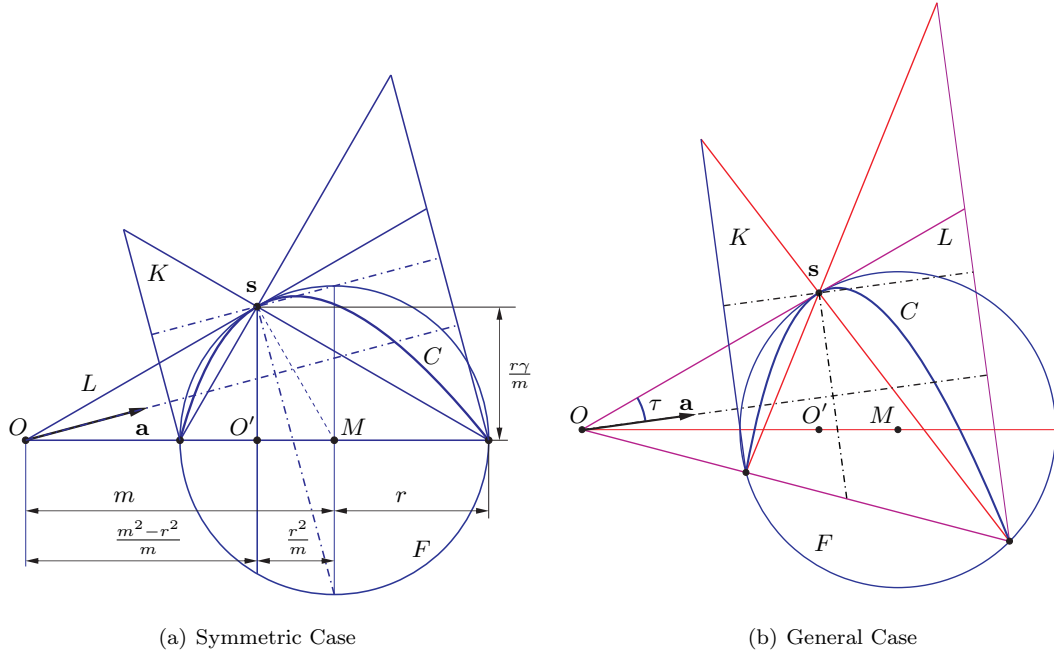


Figure 3: Geometric properties of the conchoid construction

By letting  $m \rightarrow \infty$  one obtains  $V$  as limit curve

$$\mathbf{v}(t) = (r \sin t, -r \sin t \cos t, r \cos^2 t). \quad (22)$$

Fig. 4(a) illustrates Viviani's curve  $V$ , together with the sphere and the singular quadrics belonging to the pencil. The generalized Viviani curve  $C$  being the base locus of the pencil appearing in the conchoid construction of the sphere is illustrated in Fig. 4(b). In contrast to the classical Viviani curve  $V$  whose single parameter  $r$  is the radius of the sphere  $F$ , the quartic curve  $C$  has two parameters  $r$  and  $m$ .

### 3. A family of rational spherical quartic curves with rational distance

We consider the mentioned pencil of quadrics  $Q(t) = A + tD$  from Section 2.2, and a hyperplane  $E : ax + by + cz - dw = 0$  passing through  $O = (0, 0, 0)$ . The intersection  $D \cap E$  is a quadratic cone whose projection onto  $\mathbb{R}^3$  is a cone of revolution  $L$  with axis in direction of  $\mathbf{a} = (a, b, c)$ . Assuming  $\|\mathbf{a}\| = 1$ , the opening angle  $2\tau$  of  $L$  is determined by  $d = \cos \tau$ .

Consider the quartic intersection curve  $C = F \cap L$  of a sphere  $F$  and the cone of revolution  $L$ . The quartic  $C$  is rational if it is reducible or if the cone  $L$  is touching  $F$  at a single point. Any rational curve on  $L$  has rational distance from  $O$  according to Lemma 4. Thus the quartic  $C$  is suitable to construct a rational polar representation of  $F$  if its distance from  $O$  is non-constant.

Since this touching point has to be contained in the polar plane of  $O = (0, 0, 0)$  with respect to  $F$ , we choose  $\mathbf{s} = (\gamma^2/m, 0, r\gamma/m)$  (compare 17) and prescribe an arbitrary opening angle  $2\tau$  for  $L$ . Thus the unit direction vector of  $L$ 's axis is

$$\mathbf{a}(\tau) = \frac{1}{m}(\gamma \cos \tau - r \sin \tau, 0, \gamma \sin \tau + r \cos \tau) = (a, b, c).$$

The quartic  $C$  is real if the axis is contained in the wedge formed by  $\mathbf{s}$  and the  $x$ -axis, see Figure 3(b). Thus  $-r/\gamma \leq \tan \tau \leq 0$ , because the rotation from  $\mathbf{s}$  to  $\mathbf{a}$  by  $\tau \leq 0$  is counterclockwise. The quadrics of the pencil with base locus  $C$  are denoted similarly to Section 2.4. The coefficient

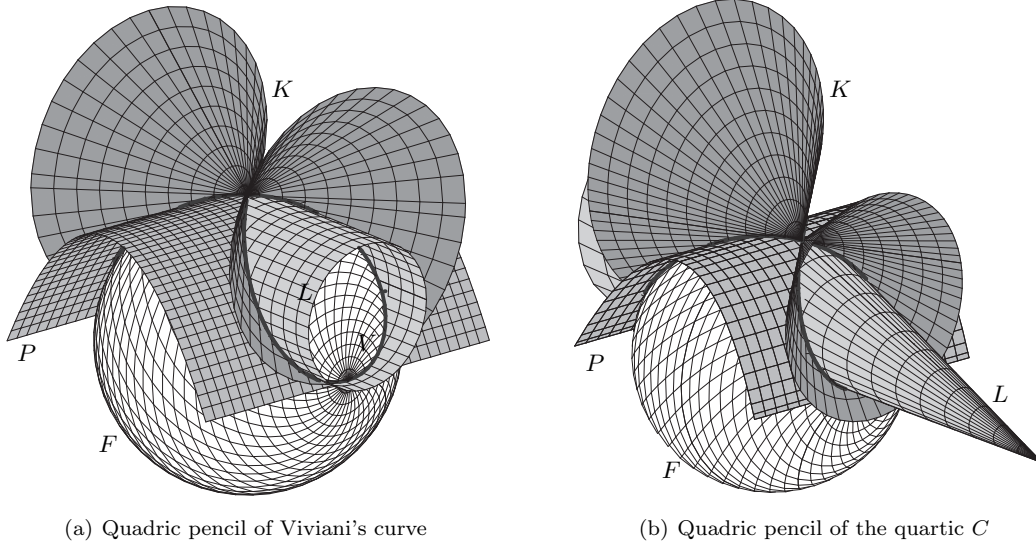


Figure 4: Quadric pencils of Viviani's curve and its generalization

matrix of the projection cone  $L$  reads

$$L(\tau) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r^2 \cos 2\tau + \gamma r \sin 2\tau & 0 & -\gamma r \cos 2\tau + \frac{1}{2}(r^2 - \gamma^2) \sin 2\tau \\ 0 & 0 & m^2 \cos^2 \tau & 0 \\ 0 & -\gamma r \cos 2\tau + \frac{1}{2}(r^2 - \gamma^2) \sin 2\tau & 0 & \gamma^2 \cos 2\tau - \gamma r \sin 2\tau \end{pmatrix}.$$

The substitutions  $\cos 2\tau = \gamma/m$ , and  $\sin 2\tau = -r/m$  yield the expression  $L$  from equation (19). This holds for all equations and parameterizations in this section in an analogous way.

The pencil of quadrics  $F + tL(\tau)$  contains two further singular quadrics corresponding to the roots

$$t_1 = \frac{-1}{m^2 \cos^2 \tau}, \text{ and } t_2 = \frac{r}{\gamma m^2 \cos \tau \sin \tau}$$

of the characteristic polynomial  $\det(F + tL(\tau))$ . The first corresponding to  $t_1$  is a parabolic cylinder  $P(\tau)$  passing through  $C$ . The  $y$ -axis is parallel to  $P$ 's generating lines, and its coefficient matrix reads

$$P(\tau) = \begin{pmatrix} \gamma^2 m^2 \cos^2 \tau & -m^3 \cos^2 \tau & 0 & 0 \\ -m^3 \cos^2 \tau & \gamma^2 \cos 2\tau + m^2 \sin^2 \tau - r\gamma \sin 2\tau & 0 & \frac{1}{2}(\gamma^2 - r^2) \sin 2\tau + r\gamma \cos 2\tau \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(\gamma^2 - r^2) \sin 2\tau + r\gamma \cos 2\tau & 0 & m^2 \cos^2 \tau - \gamma^2 \cos 2\tau + r\gamma \sin 2\tau \end{pmatrix}.$$

The second one corresponding to  $t_2$  is a cone of revolution  $K$  with axis parallel to  $\mathbf{a}$  and vertex  $\mathbf{s}$ . A parameterization of  $K$  reads  $\mathbf{k}(u, \lambda) = \mathbf{s} + \lambda(\mathbf{a} + R(\mathbf{g} \cos u + \mathbf{h} \sin u))$ , where  $\mathbf{a}$  is a unit vector in direction of its axis, and  $\mathbf{g} = (0, -1, 0)$  and  $\mathbf{h} = \mathbf{a} \times \mathbf{g}$  complete  $\mathbf{a}$  to an orthonormal basis in  $\mathbb{R}^3$ . In detail this reads

$$\mathbf{k}(u, \lambda) = \frac{1}{m} \begin{pmatrix} \gamma^2 + \lambda(\gamma \cos \tau - r \sin \tau + R \sin u(\gamma \sin \tau + r \cos \tau)) \\ -\lambda m R \cos u \\ \gamma r + \lambda(\gamma \sin \tau + r \cos \tau + R \sin u(-\gamma \cos \tau + r \sin \tau)) \end{pmatrix},$$

where  $R$  denotes the radius of the cross section circle at distance 1 from  $\mathbf{s}$ ,

$$R = \sqrt{\frac{-\sin \tau (\gamma \cos \tau - r \sin \tau)}{\cos \tau (\gamma \sin \tau + r \cos \tau)}}.$$

Note that  $R$  is *non-rational* in any rational substitution for the trigonometric functions  $\cos \tau$  and  $\sin \tau$ . The final parameterization  $\mathbf{c}(u)$  of the quartic curve  $C$  is obtained for  $\lambda = \frac{2r(R \sin u \cos \tau - \sin \tau)}{1+R^2}$  and is a bit lengthy. The incidence  $C \subset E$  implies  $\mathbf{a} \cdot \mathbf{c}(u) = w \cos \tau$ , and thus the norm of  $\mathbf{c}(u)$  reads

$$w = \|\mathbf{c}(u)\| = \frac{\gamma \cos \tau (1 + R^2) - 2r \sin \tau + 2rR \cos \tau \sin(u)}{\cos \tau (1 + R^2)}.$$

Rotating  $C$  around the  $x$ -axis gives a rational polar representation of the sphere  $F$ . This parameterization is not proper, but almost all points of  $F$  are traced twice. We summarize the construction.

**Corollary 7.** *Given a sphere  $F$  and a fixed point  $O$  there exists a one-parameter family of quartic curves  $C(\tau) \subset F$  with rational non-constant distance from  $O$ . The curves  $C(\tau)$  have common double point  $\mathbf{s}$  and  $y = 0$  as symmetry plane. The corresponding pencils of quadrics  $Q(t) = F + \lambda L(\tau)$  with base locus  $C(\tau)$  contain rotational cones  $K(\tau)$  and  $L(\tau)$ , where the vertex of the latter is at  $O$ , and a parabolic cylinder  $P(\tau)$ . Besides  $P(\tau)$  all quadrics  $Q(t)$  have rotational symmetry with axes parallel to  $\mathbf{a}(\tau)$ . The distance function  $\|\mathbf{c}(u)\|$  is rational in the curve parameter  $u$ , but not rational in the angle-parameter  $\tau$  of the family.*

#### 4. Conclusion

We have discussed the conchoid construction for spheres and have shown that a sphere  $F$  in  $\mathbb{R}^3$  admits a rational polar representation with respect to an arbitrary chosen focus point  $O$ , which implies that the conchoid surfaces of spheres possess rational parameterizations. Two constructions are presented; the first studies a pencil of quadrics in  $\mathbb{R}^4$  and constructs a parameterization of the base locus. The second construction investigates rational quartics on a sphere with rational non-constant distance from  $O$ . Relations to the classical Viviani's curve have been addressed. The first construction generalizes to rational polar representations of general quadrics in three-space. This will be elaborated in a separate contribution.

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